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1995 J. Phys. A: Math. Gen. 28 6675

(http://iopscience.iop.org/0305-4470/28/23/019)

J. Phys. A: Math. Gen. 28 (1995) 6675–6706. Printed in the UK

Bethe ansatz for higher spin eight-vertex models

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Received 2 May 1995, in final form 26 July 1995

Abstract. A generalization of the eight-vertex model by means of higher spin representations of the Sklyanin algebra is investigated by the quantum inverse scattering method and the algebraic Bethe ansatz. Under the well known string hypothesis low-lying excited states are considered and scattering phase shifts of two physical particles are calculated. The S-matrix of two-particle states is shown to be proportional to the Baxter's elliptic R-matrix with a different elliptic modulus from the original one.

Dedicated to the memory of Alexander A Belov

Introduction

In this paper we consider a generalization of the eight-vertex model by means of higher spin representations (spin ℓ) of the Sklyanin algebra [30] on a space of theta functions [31]. This model has a $(2\ell+1)$-dimensional state space on each vertical edge and a two-dimensional state space on each horizontal edge.

The relation of the eight-vertex model to the SOS type model was established by Baxter [5]. A similar relation also holds in our case and, using this relation, we can pursue the quantum inverse scattering method and the algebraic Bethe ansatz, following [24, 37, 38]. In the first part of this paper, we examine Bethe vectors and give a coordinate expression for them in terms of Boltzmann weights of SOS type model. We also prove a sum rule of rapidities of quasi-particles, which was proved for the eight-vertex model by Baxter [3] who made use of a functional equation as an alternative to the Bethe ansatz. This rule is related to the parity of Bethe vectors.

A higher spin version of the SOS type model was constructed by Date *et al* [9, 19, 8] using a fusion procedure [23, 7, 40, 17]. Recently Hasegawa [15] showed that a representation of the Sklyanin algebra obtained by a fusion procedure repeated $2\ell - 1$ times is equivalent to the spin ℓ representation on a space of theta functions [31]. Hence, in principle our model is equivalent to the higher spin SOS model developed by Date and others. The use of representations by Sklyanin [31] makes it possible to compute the eigenvectors of transfer matrices explicitly and to apply the quantum inverse scattering method and the algebraic Bethe ansatz directly (cf [6]).

As is shown by Baxter [4], the transfer matrix of the eight-vertex model contains the Hamiltonian of an anisotropic Heisenberg magnetic chain (the XYZ model) [16]. Our model is also related to a quantum spin chain model with $(2\ell+1)$-dimensional local quantum

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spaces. However, in general the transfer matrix of our lattice model does not give local Hamiltonians directly because the dimension of the auxiliary space is fixed to two. In order to write down the Hamiltonian of this spin chain, we must use the fused transfer matrix corresponding to a $(2\ell+1)$-dimensional auxiliary space. More generally we can construct a model with arbitrary spins on quantum and auxiliary spaces by fusion procedure. We will study such models in the forthcoming paper.

Note that the Bethe vectors constructed above give eigenvectors of these models simultaneously under the assumption of non-degeneracy, since transfer matrices with different auxiliary spins are mutually commuting [23].

Though momenta and Hamiltonians of spin chains are calculated from the transfer matrices of fused models, S-matrices (phase shifts) of spin waves do not depend on the auxiliary space. In the second part of this paper, we calculate a two-particle S-matrix of spin waves from Bethe vectors obtained above, following the recipe by Korepin [21], and Destri and Lowenstein [10]. The result confirms Smirnov's conjecture [12] which states that this S-matrix should be given by an elliptic R-matrix, the elliptic modulus of which is different from that of the original R-matrix in the definition of the model.

Corresponding results were established for the totally isotropic models (the XXX model) and its higher spin generalization, by Faddeev *et al* [38, 36, 2, 1] and for the XXZ model (the six-vertex model) and its higher spin generalization, by Sogo *et al* [32, 20]. The free energy of the eight-vertex model was obtained by Baxter [3] and the low-lying excited states were studied by Johnson *et al* [18], but our calculation of the S-matrix seems to be new even for the eight-vertex model ($\ell = 1/2$), though a partial result on the S-matrix for this case was calculated by Freund and Zabrodin [13]. The algebraic Bethe ansatz was shown to be applicable to the higher-spin eight-vertex models in [34] and their free energy was calculated in [35].

This paper is organized as follows. In section 1 we begin with a review of the definition of the model and the generalized algebraic Bethe ansatz, following [34]. Then, giving a whole set of intertwining vectors explicitly, we write down coordinate expressions for the Bethe vectors in terms of them. A sum rule of rapidities of quasi-particles are presented which helps in solving the Bethe equations. The proof is given in appendix B. In section 2 we study the thermodynamic limit of several Bethe vectors. The free energy is calculated in section 2.2 (this result was announced in [35]) and low-lying excited states are examined in section 2.3 under assumptions of string configurations. In particular we compute a two-particle S-matrix in section 2.3. We summarize the prerequisites for the Sklyanin algebra and its representations in appendix A.

Section 1 and appendices are algebraic in nature, while in section 2 we do not give mathematically rigorous detailed arguments, since the goal of this section is to compute quantities of physical importance. In order to make this computation rigorous strict analysis is indispensable, which is beyond the scope of this paper.

1. Description of the model and algebraic Bethe ansatz

The model was defined and its eigenvectors constructed by the generalized algebraic Bethe ansatz in [34]. In sections 1.1 and 1.3 we briefly review this work since we change notation and normalizations. In section 1.2 we find the explicit form for a whole set of intertwining vectors which enables us to write down the coordinate expression for the Bethe vectors (section 1.3). Only the 'highest' ones of intertwining vectors ('local pseudo vacua' in the context of the algebraic Bethe ansatz [37]) were used in [34] to construct Bethe vectors. In section 1.4 we show that the sum of rapidities of quasi-particles should satisfy an integrality condition. This comes from the quasi-periodicity of theta functions and is therefore absent in the case of models associated with trigonometric and rational R-matrices.

1.1. Definition of the model

The model is parametrized by a half integer ℓ and two complex parameters: an elliptic modulus τ and an anisotropy parameter η. In this paper we assume that the elliptic modulus is a pure imaginary number while the anisotropy parameter is a rational number:

$$\tau = \frac{\mathrm{i}}{t} \qquad \eta = \frac{r'}{r} \tag{1.1.1}$$

where $t > 0$ and r, r' are integers mutually coprime. Moreover we impose a condition that r is even, r' is odd, and $2(2\ell+1)\eta < 1$.

Now we define a lattice model of vertex type as in [34]. We consider a square lattice with N columns and N' rows on a torus, i.e. a periodic boundary condition imposed. States on the nth vertical edge belong to the spin ℓ representation space $V_n \simeq \Theta_{00}^{4\ell+}$ of the Sklyanin algebra (see appendix A) while states on each horizontal edge are two-dimensional vectors. A row-to-row transfer matrix, $T(\lambda)$, of the model is defined as the trace of a monodromy matrix, $\mathcal{T}(\lambda)$, in the context of the quantum inverse scattering method [37]:

$$\mathcal{T}(\lambda) = \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix} := L_N(\lambda)\dots L_2(\lambda)L_1(\lambda) \tag{1.1.2}$$

$$T(\lambda) = \mathrm{Tr}_{\mathbb{C}^2}(\mathcal{T}(\lambda)) = A_N(\lambda) + D_N(\lambda) \tag{1.1.3}$$

where the L operators, $L_n(\lambda)$, are defined by (cf (A.4))

$$\begin{aligned} L_n(\lambda) &= \begin{pmatrix} \alpha_n(\lambda) & \beta_n(\lambda) \\ \gamma_n(\lambda) & \delta_n(\lambda) \end{pmatrix} := \sum_{a=0}^{3} W_a^L(\lambda)\rho_n^\ell(S^a)\otimes\sigma^a \\ \rho_n^\ell(S^a) &= 1\otimes\dots\otimes 1\otimes\rho^\ell(S^a)\otimes 1\otimes\dots\otimes 1. \end{aligned} \tag{1.1.4}$$

Elements of these act on a Hilbert space $\mathcal{H} = \bigotimes_{n=1}^N V_n$, but non-trivially only on the nth component. Assignment of Boltzmann weights to vertices are determined by this L operator. The monodromy matrix is a 2×2 matrix with elements in $\mathrm{End}_{\mathbb{C}}(\mathcal{H})$, and the transfer matrix is an operator in $\mathrm{End}_{\mathbb{C}}(\mathcal{H})$.

The partition function, $Z(\lambda)$, and the free energy per site, $f(\lambda)$, are

$$\begin{aligned} Z(\lambda) &= \mathrm{Tr}_{\mathcal{H}}(T(\lambda)^{N'}) \\ -\beta f(\lambda) &= \frac{1}{NN'}\log Z(\lambda). \end{aligned}$$

In the thermodynamic limit, $N, N' \to \infty$, only the greatest eigenvalue, $\Lambda_{\max}$, of $T(\lambda)$ contributes to the free energy

$$-\beta f(\lambda) = \lim_{N\to\infty}\frac{1}{N}|\Lambda_{\max}| \tag{1.1.5}$$

which was computed in [35]. We will recall this result in section 2.2 with details omitted in [35].

Remark 1.1.1. The above defined model is a *homogeneous* lattice in the sense that it is invariant with respect to vertical and horizontal translation. We can also define an *inhomogeneous* lattice by assigning different spectral parameter, λ_i, and different spin, ℓ_i, to each vertical edge. We have only to replace $L_n(\lambda)$ in (1.1.2) and (1.1.3) by

$$L_n(\lambda-\lambda_n) := \sum_{a=0}^{3} W_a^L(\lambda-\lambda_n)\rho_n^{\ell_n}(S^a)\otimes\sigma^a$$

$$\rho_n^{\ell_n}(S^a) = 1 \otimes \ldots \otimes 1 \otimes \rho^{\ell_n}(S^a) \otimes 1 \otimes \ldots \otimes 1.$$

All arguments in sections 1.2 and 1.3 remain true with suitable changes, as is shown in [34]. Such models are important for the study of certain integrable systems [27].

1.2. Intertwining vectors, gauge transformation

Intertwining vectors were first introduced by Baxter [5], and given an interpretation as a gauge transformation in the context of the quantum inverse scattering method by Takhtajan and Faddeev [37]. Generalization to the higher spin case by means of fusion procedure was studied by Date *et al* [9, 8, 19]. Here we define intertwining vectors directly in the space of theta fucntions. They should be identified with those defined in [9, 8, 19], through Hasegawa's isomorphism [15].

Definition 1.2.1. Let k, k' be integers satisfying $k - k' \in \{-2\ell, -2\ell + 2, \ldots, 2\ell - 2, 2\ell\}$, and λ, $s = (s_+, s_-)$ be complex parameters. We call the following vectors $\phi_{k,k'}(\lambda; s) = \phi_{k,k'}^{(\ell)}(\lambda; s)(z) \in \Theta_{00}^{4\ell+}$ *intertwining vectors* of spin ℓ:

$$\begin{aligned}
\phi_{k,k'}^{(\ell)}(\lambda; s)(z) = a_{k,k'} \prod_{j=1}^{\ell+\frac{k-k'}{2}} & \theta\left(z + \frac{s_+ - \lambda}{2} + \frac{\tau}{4} + (k' - \ell + 2j - 1)\eta\right) \\
& \times\theta\left(z - \frac{s_+ - \lambda}{2} - \frac{\tau}{4} - (k' - \ell + 2j - 1)\eta\right) \\
& \times \prod_{j=1}^{\ell-\frac{k-k'}{2}} \theta\left(z + \frac{s_- + \lambda}{2} + \frac{\tau}{4} + (k - \ell + 2j - 1)\eta\right) \\
& \times\theta\left(z - \frac{s_- + \lambda}{2} - \frac{\tau}{4} - (k - \ell + 2j - 1)\eta\right).
\end{aligned} \tag{1.2.1}$$

Here $\theta(z) = \theta_{00}(z; \tau)$ (see (A.1)), and $a_{k,k'} = \exp[2\pi\,\mathrm{i}\ell(k + k')\eta](\mathrm{i}t^{-1/2}\exp[\pi\,\mathrm{i}(s_+ - s_-)])^{\frac{k-k'}{2}}$.

Following [37], we introduce a matrix of *gauge transformation* M_k:

$$\begin{aligned}
M_k(\lambda; s) = & \begin{pmatrix} \theta_{11}(-\mathrm{i}t(s_+ - \lambda + 2k\eta); 2\mathrm{i}t) & \theta_{11}(-\mathrm{i}t(s_- + \lambda + 2k\eta); 2\mathrm{i}t) \\ \theta_{01}(-\mathrm{i}t(s_+ - \lambda + 2k\eta); 2\mathrm{i}t) & \theta_{01}(-\mathrm{i}t(s_- + \lambda + 2k\eta); 2\mathrm{i}t) \end{pmatrix} \\
& \times \begin{pmatrix} \exp[-\frac{\pi t}{2}(s_+ - \lambda + 2k\eta - \frac{\mathrm{i}}{2t})^2] & 0 \\ 0 & \exp[-\frac{\pi t}{2}(s_- + \lambda + 2k\eta - \frac{\mathrm{i}}{2t})^2] \end{pmatrix} \\
& \times \begin{pmatrix} 1 & 0 \\ 0 & \theta_{11}(w_k; \tau)^{-1} \end{pmatrix}
\end{aligned} \tag{1.2.2}$$

where

$$w_k = \frac{s_+ + s_-}{2} + 2k\eta - \frac{\tau}{2}. \tag{1.2.3}$$

Let us define a twisted L operator by

$$\begin{aligned}
L_{k,k'}(\lambda; s) &= \begin{pmatrix} \alpha_{k,k'}(\lambda; s) & \beta_{k,k'}(\lambda; s) \\ \gamma_{k,k'}(\lambda; s) & \delta_{k,k'}(\lambda; s) \end{pmatrix} \\
&:= M_k^{-1}(\lambda; s) L(\lambda) M_{k'}(\lambda; s).
\end{aligned} \tag{1.2.4}$$

Proposition 1.2.2. Each component of $L_{k,k'}$ acts on the intertwining vector as follows.

$$
\begin{aligned}
\alpha_{k,k'}(\lambda; s)\phi_{k,k'} &= W\left(\begin{matrix} k & k' \\ k-1 & k'-1 \end{matrix}\middle|\lambda\right)\phi_{k-1,k'-1} \\
\beta_{k,k'}(\lambda; s)\phi_{k,k'} &= W\left(\begin{matrix} k & k' \\ k-1 & k'-1 \end{matrix}\middle|\lambda\right)\phi_{k-1,k'+1} \\
\gamma_{k,k'}(\lambda; s)\phi_{k,k'} &= W\left(\begin{matrix} k & k' \\ k-1 & k'-1 \end{matrix}\middle|\lambda\right)\phi_{k+1,k'-1} \\
\delta_{k,k'}(\lambda; s)\phi_{k,k'} &= W\left(\begin{matrix} k & k' \\ k-1 & k'-1 \end{matrix}\middle|\lambda\right)\phi_{k+1,k'+1}
\end{aligned}
\tag{1.2.5}
$$

where $\phi_{k,k'} = \phi^{(\ell)}_{k,k'}(0; s)$ and W is the Boltzmann weight of SOS type [5, 9]:

$$
\begin{aligned}
W\left(\begin{matrix} k & k' \\ k-1 & k'-1 \end{matrix}\middle|\lambda\right) &= 2\theta_{11}(\lambda + (k-k')\eta)\frac{\theta_{11}(w_{(k+k'+2\ell)/2})}{\theta_{11}(w_k)} \\
W\left(\begin{matrix} k & k' \\ k-1 & k'-1 \end{matrix}\middle|\lambda\right) &= 2\theta_{11}((k'-k-2\ell)\eta)\frac{\theta_{11}(w_{(k+k')/2} + \lambda)}{\theta_{11}(w_k)\theta_{11}(w_{k'})} \\
W\left(\begin{matrix} k & k' \\ k-1 & k'-1 \end{matrix}\middle|\lambda\right) &= 2\theta_{11}((k-k'-2\ell)\eta)\theta_{11}(w_{(k+k')/2} - \lambda) \\
W\left(\begin{matrix} k & k' \\ k-1 & k'-1 \end{matrix}\middle|\lambda\right) &= 2\theta_{11}(\lambda - (k-k')\eta)\frac{\theta_{11}(w_{(k+k'-2\ell)/2})}{\theta_{11}(w'_k)}.
\end{aligned}
\tag{1.2.6}
$$

This proposition is proved in the same way as (3.7) of [34].

If we denote components of $L_{k,k'}$ by

$$
L_{k,k'}(\lambda; s) = \begin{pmatrix} L_{k,k'}(-1,-1;\lambda;s) & L_{k,k'}(-1,+1;\lambda;s) \\ L_{k,k'}(+1,-1;\lambda;s) & L_{k,k'}(+1,+1;\lambda;s) \end{pmatrix}
\tag{1.2.7}
$$

(1.2.5) takes the form:

$$
L_{k,k'}(\varepsilon, \varepsilon'; \lambda; s)\phi_{k,k'} = W\left(\begin{matrix} k & k' \\ k+\varepsilon & k'+\varepsilon \end{matrix}\middle|\lambda\right)\phi_{k+\varepsilon,k'+\varepsilon'}.
\tag{1.2.8}
$$

In [34], vectors $\omega^n_m = \phi_{n+2\ell m, n+2\ell(m-1)}(s)$ were called local vacua. For these vectors the formulae (1.2.5) reduce to

$$
\alpha_{k,k-2\ell}\phi_{k,k-2\ell}(s) = 2\theta_{11}(\lambda + 2\ell\eta)\phi_{k-1,k-2\ell-1}(s) \tag{1.2.9}
$$

$$
\delta_{k,k-2\ell}\phi_{k,k-2\ell}(s) = 2\theta_{11}(\lambda - 2\ell\eta)\phi_{k+1,k-2\ell+1}(s) \tag{1.2.10}
$$

$$
\gamma_{k,k-2\ell}\phi_{k,k-2\ell}(s) = 0. \tag{1.2.11}
$$

This property is important for algebraic Bethe ansatz.

Remark 1.2.3. Denoting the column vectors of M_k by $\psi_{k,k\pm1}(\lambda; s)$, $M_k = (\psi_{k,k-1}(\lambda; s), \psi_{k,k+1}(\lambda; s))$, one can rewrite (1.2.5) as follows:

$$
\begin{aligned}
&L(\lambda - \mu)\phi^{(\ell)}_{k,k'}(\lambda; s) \otimes \psi_{k',k'+\varepsilon'}(\mu; s) \\
&\qquad = \sum_{\varepsilon} W\left(\begin{matrix} k & k' \\ k+\varepsilon & k'+\varepsilon' \end{matrix}\middle|\lambda\right)\phi^{(\ell)}_{k+\varepsilon,k'+\varepsilon'}(\lambda; s) \otimes \psi_{k,k+\varepsilon}(\mu; s).
\end{aligned}
$$

Namely ϕ and ψ intertwine the vertex weights and the SOS weights. This is where the name 'intertwining vector' comes from. See [5, 9, 14]. Note that $\phi^{(1/2)}_{k,k\pm1}(\lambda; s)$ are proportional to the column vectors of M_k under the identification (A.9).

1.3. Generalized algebraic Bethe ansatz

In this section we recall the construction of the eigenvectors of the transfer matrix by means of the algebraic Bethe ansatz, following [34], and give several properties of them. Hereafter we assume that $M := N\ell$ is an integer.

First introduce a modified monodromy matrix twisted by gauge transformation:

$$\mathcal{T}_{k,k'}(\lambda; s) = \begin{pmatrix} A_{k,k'}(\lambda; s) & B_{k,k'}(\lambda; s) \\ C_{k,k'}(\lambda; s) & D_{k,k'}(\lambda; s) \end{pmatrix} := M_k^{-1}(\lambda; s)\mathcal{T}(\lambda)M_{k'}(\lambda; s) \quad (1.3.1)$$

and *fundamental vectors* in $\mathcal{H}$:

$$|a_N, a_{N-1}, \ldots, a_1, a_0) := \phi_{a_N,a_{N-1}} \otimes \ldots \otimes \phi_{a_2,a_1} \otimes \phi_{a_1,a_0} \quad (1.3.2)$$

where $\phi_{a,b} = \phi_{a,b}(0; s)$ are intertwining vectors defined by (1.2.1). We fix a value for the parameter $s = (s_+, s_-)$ and suppress it unless it is necessary. A *pseudo vacuum*, Ω_N^a, is a fundamental vector characterized by $a_0 = a$, $a_i - a_{i-1} = 2\ell$ for all $i = 1, \ldots, N$:

$$\Omega_N^a = |a + 2N\ell, a + 2(N-1)\ell, \ldots, a + 2\ell, a). \quad (1.3.3)$$

This vector satisfies

$$A_{a+2N\ell,a}(\lambda)\Omega_N^a = (2\theta_{11}(\lambda + 2\ell\eta))^N \Omega_N^{a-1} \quad (1.3.4)$$
$$D_{a+2N\ell,a}(\lambda)\Omega_N^a = (2\theta_{11}(\lambda - 2\ell\eta))^N \Omega_N^{a+1} \quad (1.3.5)$$
$$C_{a+2N\ell,a}(\lambda)\Omega_N^a = 0 \quad (1.3.6)$$

by virtue of (1.2.9)–(1.2.11), respectively.

As is shown in [34], the *algebraic Bethe ansatz* for our case leads to the following:

Proposition 1.3.1. Let ν be an integer, $\lambda_1, \ldots, \lambda_M$ complex numbers. Define a vector $\Psi_\nu(\lambda_1, \ldots, \lambda_M) \in \mathcal{H}$ by

$$\Psi_\nu(\lambda_1, \ldots, \lambda_M) := \sum_{a=0}^{r-1} \mathrm{e}^{2\pi\,\mathrm{i}\nu\eta a} \Phi_a(\lambda_1, \ldots, \lambda_M)$$
$$\Phi_a(\lambda_1, \ldots, \lambda_M) := B_{a+1,a-1}(\lambda_1)B_{a+2,a-2}(\lambda_2) \ldots B_{a+M,a-M}(\lambda_M)\Omega_N^{a-M}. \quad (1.3.7)$$

Then $\Psi_\nu(\lambda_1, \ldots, \lambda_M)$ is an eigenvector of the transfer matrix $T(\lambda)$ with an eigenvalue

$$t(\lambda) = \mathrm{e}^{2\pi\,\mathrm{i}\nu\eta}(2\theta_{11}(\lambda + 2\ell\eta; \tau))^N \prod_{j=1}^{M} \frac{\theta_{11}(\lambda - \lambda_j - 2\eta; \tau)}{\theta_{11}(\lambda - \lambda_j; \tau)}$$
$$+ \mathrm{e}^{-2\pi\,\mathrm{i}\nu\eta}(2\theta_{11}(\lambda - 2\ell\eta; \tau))^N \prod_{j=1}^{M} \frac{\theta_{11}(\lambda - \lambda_j + 2\eta; \tau)}{\theta_{11}(\lambda - \lambda_j; \tau)} \quad (1.3.8)$$

provided that ν and $\{\lambda_1, \ldots, \lambda_M\}$ satisfy the following *Bethe equations*:

$$\left(\frac{\theta_{11}(\lambda_j + 2\ell\eta; \tau)}{\theta_{11}(\lambda_j - 2\ell\eta; \tau)}\right)^N = \mathrm{e}^{-4\pi\,\mathrm{i}\nu\eta} \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{\theta_{11}(\lambda_j - \lambda_k + 2\eta; \tau)}{\theta_{11}(\lambda_j - \lambda_k - 2\eta; \tau)} \quad (1.3.9)$$

for all $j = 1, \ldots, M$.

Proof. The proof is the same as that in [37]. We only recall the periodicity of the vector Φ_a with respect to a, which is the reason that we do not have to take an infinite sum in the definition of Ψ_ν.

Recall that η is a rational number r'/r. Therefore $2(k+r)\eta \equiv 2k\eta$ (mod 2). This fact and quasi-periodicity of theta functions imply $M_{k+r} = M_k$. Hence (see (1.3.1))

$$\mathcal{T}_{k+r,k'+r}(\lambda) = \mathcal{T}_{k,k'}(\lambda) \tag{1.3.10}$$

in particular, $B_{k+r,k'+r}(\lambda) = B_{k,k'}(\lambda)$. Similarly one can prove $\phi_{k+r,k'+r} = \phi_{k,k'}$. This proves $\Phi_{a+r}(\lambda_1, \ldots, \lambda_M) = \Phi_a(\lambda_1, \ldots, \lambda_M)$. □

The eigenvalue (1.3.8) is written in a compact form in terms of a function $Q(\lambda)$ defined by

$$Q(\lambda) = \mathrm{e}^{-\pi \mathrm{i}\nu\lambda} \prod_{j=1}^{M} \theta_{11}(\lambda - \lambda_j). \tag{1.3.11}$$

The eigenvalue of the transfer matrix for a Bethe vector $\Psi_\nu(\lambda_1, \ldots, \lambda_M)$ is

$$t(\lambda) := h(\lambda + 2\ell\eta)\frac{Q(\lambda - 2\eta)}{Q(\lambda)} + h(\lambda - 2\ell\eta)\frac{Q(\lambda + 2\eta)}{Q(\lambda)} \tag{1.3.12}$$

where $h(z) = (2\theta_{11}(z))^N$. The Bethe equations (1.3.9) can be interpreted as the condition of cancellation of poles at λ_j of the right-hand side of the above equation. This observation is due to Baxter [3] and a starting point of Reshetikhin's analytic Bethe ansatz [28]. We essentially use this observation to derive the sum rule in section 1.4.

Because of the commutation relation $B_{k,k'+1}(\lambda)B_{k+1,k'}(\mu) = B_{k,k'+1}(\mu)B_{k+1,k'}(\lambda)$, which is a consequence of relation (A.3), $\Psi_\nu(\lambda_1, \ldots, \lambda_M)$ does not depend on the order of parameters $\lambda_1, \ldots, \lambda_M$. Moreover, we may restrict the parameters to the fundamental domain

$$-\tfrac{1}{2} \leqslant \operatorname{Re}\lambda_j \leqslant \tfrac{1}{2} \qquad -\frac{\tau}{2} \leqslant \operatorname{Im}\lambda_j \leqslant \frac{\tau}{2} \tag{1.3.13}$$

without loss of generality thanks to the following lemma.

Lemma 1.3.2. Suppose $(\nu, \{\lambda_1, \ldots, \lambda_M\})$ is a solution of the Bethe equations. Then for any j, $1 \leqq j \leqq M$,

(i) $\Psi_\nu(\lambda_1, \ldots, \lambda_j + 1, \ldots, \lambda_M)$ is proportional to $\Psi_\nu(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_M)$, and $(\nu, \{\lambda_1, \ldots, \lambda_j + 1, \ldots, \lambda_M\})$ is a solution of the Bethe equations.

(ii) $\Psi_{\nu+2}(\lambda_1, \ldots, \lambda_j + \tau, \ldots, \lambda_M)$ is proportional to $\Psi_\nu(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_M)$, and $(\nu + 2, \{\lambda_1, \ldots, \lambda_j + \tau, \ldots, \lambda_M\})$ is a solution of the Bethe equations.

Proof. The quasi-periodicity of theta functions implies that $(\nu, \{\lambda_1, \ldots, \lambda_j + 1, \ldots, \lambda_M\})$ and $(\nu + 2, \{\lambda_1, \ldots, \lambda_j + \tau, \ldots, \lambda_M\})$ are solutions of the Bethe equations.

Substituting $\lambda \to \lambda + 1$ and $\lambda \to \lambda + \tau$ in (1.1.4), we obtain

$$\begin{aligned} L_n(\lambda + 1) &= -\sigma^1 L_n(\lambda)\sigma^1 \\ L_n(\lambda + \tau) &= -\mathrm{e}^{-\pi \mathrm{i}\tau - 2\pi \mathrm{i}\lambda}\sigma^3 L_n(\lambda)\sigma^3. \end{aligned}$$

In the same way (see (1.2.2))

$$\begin{aligned} M_k(\lambda + 1; s) &= \sigma^1 M_k(\lambda; s) \\ M_k(\lambda + \tau; s) &= \sigma^3 M_k(\lambda; s)\begin{pmatrix} -\mathrm{e}^{\pi \mathrm{i}(s_+ + 2k\eta - \lambda - \tau)} & 0 \\ 0 & -\mathrm{e}^{-\pi \mathrm{i}(s_- + 2k\eta + \lambda)} \end{pmatrix}. \end{aligned}$$

Hence from (1.3.1) follows

$$\mathcal{T}_{k,k'}(\lambda+1;s) = (-1)^N \mathcal{T}_{k,k'}(\lambda) \tag{1.3.14}$$

$$\mathcal{T}_{k,k'}(\lambda+\tau;s) = (-1)^N e^{\pi\,\mathrm{i}N\tau - 2\pi\,\mathrm{i}N\lambda} \begin{pmatrix} e^{-\pi\,\mathrm{i}(s_+ + 2k\eta - \lambda - \tau)} & 0 \\ 0 & e^{\pi\,\mathrm{i}(s_- + 2k\eta + \lambda)} \end{pmatrix}$$
$$\times \mathcal{T}_{k,k'}(\lambda;s) \begin{pmatrix} e^{\pi\,\mathrm{i}(s_+ + 2k'\eta - \lambda - \tau)} & 0 \\ 0 & e^{-\pi\,\mathrm{i}(s_- + 2k'\eta + \lambda)} \end{pmatrix}. \tag{1.3.15}$$

The (1,2)-component of (1.3.15) is $B_{k,k'}(\lambda+1) = (-1)^N B_{k,k'}(\lambda)$. Thus

$$\Psi_\nu(\lambda_1,\ldots,\lambda_j+1,\ldots,\lambda_M) = (-1)^N \Psi_\nu(\lambda_1,\ldots,\lambda_j,\ldots,\lambda_M)$$

which proves (i). The (1,2)-component of (1.3.15) is

$$B_{k,k'}(\lambda+\tau) = \text{constant } e^{-2\pi\,\mathrm{i}(k+k')\eta} B_{k,k'}(\lambda).$$

Here constant does not depend on k, k', but depends on λ, $s_\pm$. Thus

$$\Phi_a(\lambda_1,\ldots,\lambda_j+\tau,\ldots,\lambda_M) = \text{constant } e^{-4\pi\,\mathrm{i}a\eta} \Phi_a(\lambda_1,\ldots,\lambda_j,\ldots,\lambda_M)$$

and

$$\begin{aligned}\Psi_{\nu+2}(\lambda_1,\ldots,\lambda_j+\tau,\ldots,\lambda_M) &= \text{constant } \sum_{a=0}^{r-1} e^{2\pi\,\mathrm{i}(\nu+2)a\eta - 4\pi\,\mathrm{i}a\eta} \Phi_a(\lambda_1,\ldots,\lambda_M) \\ &= \text{constant } \Psi_\nu(\lambda_1,\lambda_2,\ldots,\lambda_M).\end{aligned}$$

This proves (ii). □

Baxter developed the coordinate Bethe ansatz in [5], expanding eigenvectors into linear combination of fundamental vectors $|a_0,\ldots,a_N\rangle$. We have such a coordinate expression for the above defined Bethe vectors. Namely

Proposition 1.3.3. Let $\Psi_\nu(\lambda_1,\ldots,\lambda_M)$ be as defined by (1.3.7). Then

$$\Psi_\nu(\lambda_1,\ldots,\lambda_M) = \sum_{a=0}^{r-1} e^{2\pi\,\mathrm{i}\nu\eta a}$$
$$\times \sum_{\substack{a_0=a,a_1,\ldots \\ \ldots,a_{N-1},a_N=a}} \left(\sum_{\{a_{i,j}\}} \prod_{i=1}^{M} \prod_{j=1}^{N} W\left(\begin{matrix} a_{i,j} & a_{i,j-1} \\ a_{i-1,j} & a_{i-1,j-1} \end{matrix} \middle| \lambda_i \right) \right) |a_N,\ldots,a_1,a_0\rangle. \tag{1.3.16}$$

Here the sum in () is taken over a set of integers $a_{i,j}$ $(0 \leqq i \leqq M,\ 0 \leqq j \leqq N)$ satisfying *the admissibility condition*

$$a_{i,j} - a_{i-1,j} = \pm 1 \qquad a_{i,j} - a_{i,j-1} \in \{-2\ell, -2\ell+2, \ldots, 2\ell-2, 2\ell\}$$

and *the boundary condition,*

$$\begin{aligned} a_{0,j} &= a_j & a_{M,j} &= a - N\ell + 2\ell j \\ a_{i,0} &= a - i & a_{i,N} &= a + i. \end{aligned}$$

Note that the Bethe equations are not assumed here.

Proof. Operator $B_{a_N,a_0}(\lambda)$ is the $(1,2)$-element of the monodromy matrix $\mathcal{T}_{a_N,a_0}(\lambda) = L_{a_N,a_{N-1}} \dots L_{a_1,a_0}$. Hence by (1.2.8),

$$
\begin{aligned}
B_{a_N,a_0}(\lambda)|a_N, a_{N-1}, \dots, a_1, a_0\rangle
&= \sum_{\substack{\varepsilon_N=-,\varepsilon_{N-1},\dots\\ \dots,\varepsilon_1,\varepsilon_0=+}} L_{a_N,a_{N-1}}(\varepsilon_N, \varepsilon_{N-1}; \lambda) \dots L_{a_j,a_{j-1}}(\varepsilon_j, \varepsilon_{j-1}; \lambda) \dots \\
&\quad \dots L_{a_1,a_0}(\varepsilon_1, \varepsilon_0; \lambda) \phi_{a_N,a_{N-1}} \otimes \dots \otimes \phi_{a_j,a_{j-1}} \otimes \dots \otimes \phi_{a_1,a_0} \\
&= \sum_{\substack{\varepsilon_N=-,\varepsilon_{N-1},\dots\\ \dots,\varepsilon_1,\varepsilon_0=+}} \prod_{j=1}^{N} W\left(\begin{matrix} a_j & a_{j-1} \\ a_j+\varepsilon_j & a_{j-1}+\varepsilon_{j-1} \end{matrix} \middle| \lambda \right) \phi_{a_N+\varepsilon_N,a_{N-1}+\varepsilon_{N-1}} \otimes \dots \otimes \\
&\quad \times \phi_{a_j+\varepsilon_j,a_{j-1}+\varepsilon_{j-1}} \otimes \dots \otimes \phi_{a_1+\varepsilon_1,a_0+\varepsilon_0}.
\end{aligned}
$$

Applying this formula iteratively, we arrive at (1.3.16). □

1.4. Sum rule

In the previous section we defined Bethe vectors by (1.3.7). Here we show an integrality condition of sum of parameters λ_j.

Theorem 1.4.1. Let $(\nu, \{\lambda_1, \dots, \lambda_M\})$ be a solution of the Bethe equations (1.3.9). We assume that $\{\lambda_j\}$ satisfy the following additional conditions. For any $j = 1, \dots, M$,

(i) $\lambda_j \notin \{2(n+\ell)\eta \mid n \in \mathbb{Z}\}$;

(ii) there exists $a \in \mathbb{Z}$ such that $\lambda_j + 2a\eta \not\equiv \lambda_k \pmod{\mathbb{Z}+\mathbb{Z}\tau}$ for any $k = 1, \dots, M$.

(iii) (Technical assumption of non-degeneracy: see appendix B.)

Then there exist integers n_0, n_1 which satisfy

$$2\sum_{j=1}^{M} \lambda_j = n_0 + n_1\tau. \tag{1.4.1}$$

The proof is technical and contained in appendix B. Note that assumptions (i) and (ii) in theorem 1.4.1 are satisfied under the string hypothesis in section 2.1. Assumption (iii) is hard to check. For string solutions considered in sections 2.2 and 2.3, (1.4.1) is checked directly.

Baxter derived this rule in [3], directly constructing an operator on $\mathcal{H}$ which gives $Q(\lambda)$ (see (1.3.11)) as its eigenvalue. Unfortunately we have not yet found such an operator in our context. (Kulish and Reshetikhin [22] found for a rational R matrix case that transfer matrix 'converges' to the Q operator by iteration of fusion procedures.)

In addition, Baxter's result tells us that $\sum_{j=1}^{M} \lambda_j$ is related to parities of Bethe vectors. These parities are associated with reversing the arrows (σ^1) or to assigning -1 to the down arrows (σ^3) of the XYZ spin chain model. They correspond to the unitary operators U_1 and U_3 acting on the spin ℓ representations (see appendix A).

Lemma 1.4.2. For $a = 1, 2, 3$, U_a commutes with the L operator as

$$U_a^{-1} L(\lambda) U_a = (\sigma^a)^{-1} L(\lambda) \sigma^a. \tag{1.4.2}$$

They commute with the transfer matrix: $U_a^{-1} T(\lambda) U_a = T(\lambda)$.

Proof. Adjoint action by U_a induces an automorphism X_a on the Sklyanin algebra (see appendix A):

$$\begin{aligned} U_a^{-1} L(\lambda) U_a &= X_a(L(\lambda)) \\ &= W_0(\lambda)\rho^\ell(S^0)\otimes\sigma^0 + W_a(\lambda)\rho^\ell(S^a)\otimes\sigma^a \\ &\quad - W_b(\lambda)\rho^\ell(S^b)\otimes\sigma^b - W_c(\lambda)\rho^\ell(S^c)\otimes\sigma^c \end{aligned}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. By the anti-commutativity of the Pauli matrices, the right-hand side of the above equation is nothing but $(\sigma^a)^{-1} L(\lambda)\sigma^a$.

Commutativity $T(\lambda)U_a = U_a T(\lambda)$ is a direct consequence of (1.4.2) and (1.1.3). □

Operators $U_a^{\otimes N}$ on $\mathcal{H}$ are involutive and commute with each other. In fact, by virtue of relations $U_a^2 = (-1)^{2\ell}$ and $U_a U_b = (-1)^{2\ell} U_b U_a$, we have

$$\begin{aligned} &(U_a^{\otimes N})^2 = (-1)^{2N\ell} = 1 \\ &U_a^{\otimes N} U_b^{\otimes N} = (-1)^{2N\ell} U_b^{\otimes N} U_a^{\otimes N} = U_b^{\otimes N} U_a^{\otimes N}. \end{aligned}$$

(Recall that $N\ell$ is an integer.) Therefore an eigenvalue of $U_a^{\otimes N}$ is either $+1$ or -1. Assume that $\Psi \in \mathcal{H}$ is a common eigenvector of $T(\lambda)$ and U_a's. We assign parities ν'' and ν' to Ψ by

$$U_1^{\otimes N}\Psi = (-1)^{\nu''}\Psi \qquad U_3^{\otimes N}\Psi = (-1)^{\nu'}\Psi.$$

From Baxter's result [3] and theorem 1.4.1, the following conjecture seems to be plausible.

Conjecture 1.4.3. Let $(\nu, \{\lambda_1, \ldots, \lambda_M\})$ be a solution of the Bethe equations, and ν'' and ν' be parities of the Bethe vector $\Psi_\nu(\lambda_1, \ldots, \lambda_M)$ defined above. Then

$$\begin{aligned} &\nu + \nu' + N\ell \equiv 0 \qquad (\mathrm{mod}\ 2) \\ &\nu\tau - 2\sum_{j=1}^{M}\lambda_j \equiv \nu'' + N\ell \qquad (\mathrm{mod}\ 2). \end{aligned}$$

2. Thermodynamic limit

In this chapter we consider the limit $N \to \infty$. Our calculation is based on the string hypothesis introduced by Takahashi and Suzuki [33] which we review in section 2.1. In section 2.2 we compute the free energy of the model. In section 2.3 we introduce four kinds of perturbation of the string configuration of the ground state. Each of them are parametrized by two continuous parameters which are regarded as rapidities of physical particles. We compute polarization of the Dirac sea of quasi-particles induced by these perturbation, following the recipe in Johnson *et al* [18]. We also calculate eigenvalues of the S-matrix of two physical particles, using the method developed by Korepin [21] and Destri and Lowenstein [10]. The result coincides with Smirnov's conjecture [12].

2.1. String hypothesis

First let us rescale the parameters so that integrals in later sections are taken over a segment in the real line. We denote $x_j = it\lambda_j$. Then the Bethe equations (1.3.9) take the form

$$\left(\frac{\theta_{11}(x_j + 2i\ell\eta t; it)}{\theta_{11}(x_j - 2i\ell\eta t; it)}\right)^N = e^{-4\pi i\eta(\nu + 2\sum_{k=1}^{M} x_k)} \prod_{\substack{k=1 \\ k\neq j}}^{M} \frac{\theta_{11}(x_j - x_k + 2i\eta t; it)}{\theta_{11}(x_j - x_k - 2i\eta t; it)} \tag{2.1.1}$$

while the corresponding eigenvalue is

$$
\begin{aligned}
t(x) = (-2\mathrm{i}\sqrt{t})^N \exp\left[\frac{\pi N}{t}(x^2 - 4\ell(\ell+1)\eta^2 t^2)\right] \Bigg(& \exp\left[2\pi\,\mathrm{i}\eta\left(\nu + 2\sum_{j=1}^{M} x_j\right)\right] \\
& \times (2\theta_{11}(x + 2\mathrm{i}\ell\eta t; \mathrm{i}t))^N \prod_{j=1}^{M} \frac{\theta_{11}(x - x_j - 2\mathrm{i}\eta t; \mathrm{i}t)}{\theta_{11}(x - x_j; \mathrm{i}t)} \\
& + \exp\left[-2\pi\,\mathrm{i}\eta\left(\nu + 2\sum_{j=1}^{M} x_j\right)\right] (2\theta_{11}(x - 2\mathrm{i}\ell\eta t; \mathrm{i}t))^N \\
& \times \prod_{j=1}^{M} \frac{\theta_{11}(x - x_j + 2\mathrm{i}\eta t; \mathrm{i}t)}{\theta_{11}(x - x_j; \mathrm{i}t)} \Bigg)
\end{aligned}
\tag{2.1.2}
$$

where $x = \mathrm{i}t\lambda$. According to lemma 1.3.2, we may assume that $|\mathrm{Re}(x_j)| \leqslant 1/2$ for any j.

Now *the string hypothesis* [33] is stated in the following way. For sufficiently large N solutions of (2.1.1) cluster into groups known as A-strings, $A = 1, 2, \ldots$, with parity $\pm$:

$$
x_{j,\alpha}^{A,\pm} = x_j^{A,\pm} + 2\mathrm{i}\eta t\alpha + O(\mathrm{e}^{-\delta N}) \qquad \alpha = \frac{-A+1}{2}, \frac{-A+3}{2}, \ldots, \frac{A-1}{2} \tag{2.1.3}
$$

where $\mathrm{Im} x_j^{A,+} \equiv 0 \pmod{\mathbb{Z}t}$ and $\mathrm{Im} x_j^{A,-} \equiv t/2 \pmod{\mathbb{Z}t}$. Complex numbers $x_j^{A,\pm}$ are called a centre of a string. Due to lemma 1.3.2 we may assume that $\mathrm{Im} x_j^{A,+} = 0$ and $\mathrm{Im} x_j^{A,-} = t/2$. We denote the number of A-strings with parity $\pm$ by $\sharp(A, \pm)$.

Note that assumption (i) of theorem 1.4.1 is satisfied for A-strings with parity $+$, if $A \equiv 2\ell \pmod 2$ and for any strings with non-zero real abscissa. Assumption (ii) holds for A-strings, $A < r$, if real parts of centres of all strings lie in the interval $[-1/2, 1/2]$.

Since we are interested only in the thermodynamic limit $N \to \infty$, we neglect the exponentially small deviation term $O(\mathrm{e}^{-\delta N})$ in what follows.

2.2. *Ground state and free energy*

The result in this section was announced in [35]. The ground state configuration is specified as follows: $\nu = 0$, $\sharp(2\ell, +) = N/2$, $\sharp(A, \pm) = \sharp(2\ell, -) = 0$ for $A \neq 2\ell$ and centres of 2ℓ-strings distribute symmetrically around 0. (Hence $\sum_{j=1}^{N/2} x_j^{2\ell,+} = 0$.) This is consistent with the result of the XXX type model by Takhtajan [36], Babujian [2], that of the XXZ type by Sogo [32], Kirillov and Reshetikhin [20] and that of the XYZ model by Baxter [3].

Multiplying the Bethe equations (2.1.1) for $x_j = x_{j,\alpha}^{2\ell,+}$, $\alpha = -\ell + 1/2, \ldots, \ell - 1/2$ (cf (2.1.3)), and taking the logarithm, we obtain

$$
\begin{aligned}
& N \sum_{\alpha=-\ell+1/2}^{\ell-1/2} \Phi(x_j; 2\mathrm{i}(\alpha+\ell)\eta t) \\
& \quad = 2\pi Q_j^{2\ell} + \sum_{k=1}^{N/2} \left(\sum_{m=1}^{2\ell-1} \Phi(x_j - x_k; 2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1} \Phi(x_j - x_k; 2\mathrm{i}(m+1)\eta t) \right)
\end{aligned}
\tag{2.2.1}
$$

where we omit the index $(2\ell, +)$ of $x_j^{2\ell,+}$ and function Φ is defined by

$$
\Phi(x; \mathrm{i}\mu t) = \frac{1}{i} \log \frac{\theta_{11}(x + \mathrm{i}\mu t; \mathrm{i}t)}{\theta_{11}(x - \mathrm{i}\mu t; \mathrm{i}t)} + \pi. \tag{2.2.2}
$$

We take the branch of Φ so that $\Phi(\pm 1/2; \mathrm{i}\mu) = \mp\pi$, $\Phi(0; \mathrm{i}\mu) = 0$. Half integers $Q_j^{2\ell}$ in (2.2.1) specify the branches of logarithm. Applying Takhtajan–Faddeev's argument [38] to our case, they satisfy $Q_j^{2\ell} - Q_{j-1}^{2\ell} = -1$. We also assume that the x_j are ordered by j: $x_j > x_{j-1}$. (Note that $\Phi(x; \mathrm{i}\mu t)$ is a decreasing function by lemma C.2.)

We assume that centres of 2ℓ strings fill in the interval $(-1/2, 1/2)$ with density $\rho(x)$ in the limit $N \to \infty$,

$$\frac{1}{N(x_{j+1} - x_j)} \to \rho(x) \qquad x_j \to x \qquad N \to \infty.$$

Subtracting (2.2.1) for j from that for $j+1$ and taking the limit, we obtain an integral equation

$$\begin{aligned}\sum_{\alpha=-\ell+1/2}^{\ell-1/2} \Phi'(x; 2\mathrm{i}(\alpha+\ell)\eta t) &= -2\pi\rho(x) \\ &+ \int_{-1/2}^{1/2} \left(\sum_{m=1}^{2\ell-1} \Phi'(x-y; 2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1} \Phi'(x-y; 2\mathrm{i}(m+1)\eta t) \right) \rho(y)\, \mathrm{d}y.\end{aligned} \tag{2.2.3}$$

This equation is easily solved by Fourier expansion. Using formula (C.2), we have

$$\rho(x) = \sum_{n\in\mathbb{Z}} \frac{\mathrm{e}^{2\pi\,\mathrm{i}nx}}{2\cos h 2\pi n\eta t}. \tag{2.2.4}$$

Let us compute the eigenvalue for this Bethe vector. The expression of the eigenvalue (2.1.2) consists of two terms:

$$\begin{aligned}\Lambda_1(x) &= (-2\mathrm{i}\sqrt{t})^N \exp\left[\frac{\pi N}{t}(x^2 - 4\ell(\ell+1)\eta^2 t^2)\right] \\ &\quad \times (2\theta_{11}(x + 2\mathrm{i}\ell\eta t; \mathrm{i}t))^N \prod_{j=1}^{M} \frac{\theta_{11}(x - x_j - 2\mathrm{i}\eta t; \mathrm{i}t)}{\theta_{11}(x - x_j; \mathrm{i}t)} \\ \Lambda_2(x) &= (-2\mathrm{i}\sqrt{t})^N \exp\left[\frac{\pi N}{t}(x^2 - 4\ell(\ell+1)\eta^2 t^2)\right] \\ &\quad \times (2\theta_{11}(x - 2\mathrm{i}\ell\eta t; \mathrm{i}t))^N \prod_{j=1}^{M} \frac{\theta_{11}(x - x_j + 2\mathrm{i}\eta t; \mathrm{i}t)}{\theta_{11}(x - x_j; \mathrm{i}t)}\end{aligned} \tag{2.2.5}$$

and the eigenvalue is $t(x) = \Lambda_1(x) + \Lambda_2(x)$. Both Λ_1 and Λ_2 contribute equally to $t(x)$ in the thermodynamic limit $N \to \infty$ for $\ell > 1/2$, since

$$\begin{aligned}\frac{1}{N}\log\frac{\Lambda_1}{\Lambda_2} &= \log\frac{\theta_{11}(x + 2\mathrm{i}\ell\eta t; \mathrm{i}t)}{\theta_{11}(x - 2\mathrm{i}\ell\eta t : \mathrm{i}t)} \\ &\quad + \frac{1}{N}\sum_{j=1}^{N/2}\left(\log\frac{\theta_{11}(x - x_j - \mathrm{i}(2\ell+1)\eta t; \mathrm{i}t)}{\theta_{11}(x - x_j + \mathrm{i}(2\ell+1)\eta t; \mathrm{i}t)}\right. \\ &\quad \left. + \log\frac{\theta_{11}(x - x_j - \mathrm{i}(2\ell-1)\eta t; \mathrm{i}t)}{\theta_{11}(x - x_j + \mathrm{i}(2\ell-1)\eta t; \mathrm{i}t)}\right) \\ &\quad \xrightarrow{N\to\infty} \log\frac{\theta_{11}(x + 2\mathrm{i}\ell\eta t; \mathrm{i}t)}{\theta_{11}(x - 2\mathrm{i}\ell\eta t : \mathrm{i}t)} \\ &\quad + \int_{-1/2}^{1/2}\left(\log\frac{\theta_{11}(x - y - \mathrm{i}(2\ell+1)\eta t; \mathrm{i}t)}{\theta_{11}(x - y + \mathrm{i}(2\ell+1)\eta t; \mathrm{i}t)}\right. \\ &\quad \left. + \log\frac{\theta_{11}(x - y - \mathrm{i}(2\ell-1)\eta t; \mathrm{i}t)}{\theta_{11}(x - y + \mathrm{i}(2\ell-1)\eta t; \mathrm{i}t)}\right)\rho(y)\,\mathrm{d}y = -2\pi\mathrm{i}.\end{aligned}$$

In the case of the eight-vertex model ($\ell = 1/2$) for $\lambda > 0$, Λ_1 is dominant in magnitude, because

$$\frac{1}{N}\log\frac{\Lambda_1}{\Lambda_2} = \log\frac{\theta_{11}(x+\mathrm{i}\eta t;\mathrm{i}t)}{\theta_{11}(x-\mathrm{i}\eta t:\mathrm{i}t)} + \frac{1}{N}\sum_{j=1}^{N/2}\log\frac{\theta_{11}(x-x_j-2\mathrm{i}\eta t;\mathrm{i}t)}{\theta_{11}(x-x_j+2\mathrm{i}\eta t;\mathrm{i}t)}$$
$$\xrightarrow{N\to\infty} -\frac{3\pi\mathrm{i}}{2} - \pi\,\mathrm{i}x - \sum_{n=1}^{\infty}\frac{\mathrm{i}\sin 2\pi nx}{n\cos h2\pi n\eta t} \tag{2.2.6}$$

the real part of which is positive. (Recall that $x = \mathrm{i}t\lambda$, $\lambda > 0$.) This is a subtle difference between $\ell = 1/2$ and higher spin cases, but the final result does not differ much. Namely, in the thermodynamic limit,

$$\frac{1}{N}\log t(x) \overset{N\to\infty}{\longrightarrow} \frac{1}{N}\log\Lambda_1(x)$$
$$\overset{N\to\infty}{\sim} \log(-2\mathrm{i}) + \frac{1}{2}\log t + \frac{\pi}{t}(x^2 - 4\ell(\ell+1)\eta^2t^2) + \log 2 +$$
$$+\log\theta_{11}(x+2\mathrm{i}\ell\eta t;\mathrm{i}t) + \int_{-1/2}^{1/2}\log\frac{\theta_{11}(x-y-2\mathrm{i}\eta t;\mathrm{i}t)}{\theta_{11}(x-y;\mathrm{i}t)}\rho(y)\,\mathrm{d}y. \tag{2.2.7}$$

Substituting (2.2.4) into this, we obtain the free energy (1.1.5):

$$-\beta f(\lambda) = (\text{constant}) + \log\theta_{11}(\lambda+2\ell\eta;\tau) - 2\pi t(\lambda-\eta)(1-4\ell\eta)$$
$$-\sum_{n=1}^{\infty}\frac{\sin h\pi nt(1-4\ell\eta)\,\sin h2\pi nt(\lambda-\eta)}{n\sin h\pi nt\,\cos h2\pi n\eta t}. \tag{2.2.8}$$

Here (constant) is an unessential term which does not depend on λ.

2.3. *Low-lying excitations and S-matrices*

As is seen in section 2.2, the ground state consists of $N/2$ 2ℓ-strings filling the Dirac sea. In this section we perturb this Dirac sea, slightly changing the string configuration. Since we are interested in the two-particle states, we choose such configurations that reduce to two-particle states of models associated with trigonometric and rational R-matrices [38, 36, 20, 2, 32].

Let us consider the following configurations:

(I) $\sharp(2\ell,+) = N/2-2$, $\sharp(2\ell-1,+) = 1$, $\sharp(2\ell+1,+) = 1$
(II) $\sharp(2\ell,+) = N/2-1$, $\sharp(2\ell-1,+) = 1$, $\sharp(1,-) = 1$.

We call the Bethe vectors specified by these data *excited state* I and II respectively. In the higher spin XXX case [36], for example, two-particle states are specified by similar configurations; one (singlet) is the same as I above, the other (triplet) is defined by $\sharp(2\ell,+) = N/2-1$, $\sharp(2\ell-1,+) = 1$. Since one-string with parity $-$ goes to infinity when t tends to ∞, we can expect that excited state II reduces to the triplet state in the rational limit. As a matter of course, when $\ell = 1/2$, the $(2\ell-1)$-string is absent. Hence the following argument needs to be modified, but one obtains results for $\ell = 1/2$ by simply putting $\ell = 1/2$ in formulae for general ℓ. We do not mention this modification.

2.3.1. *Excited state I.*

Now we consider the excited state I. We omit the plus sign designating the parity, since all strings have parity $+$. Multiplying the Bethe equations (2.2.1) for a 2ℓ- string $x_j = x_{j,\alpha}^{2\ell}$, $\alpha = -\ell+1/2,\ldots,\ell-1/2$ with the centre $x_j^{2\ell}$, and taking the logarithm, we obtain

$$
\begin{aligned}
N \sum_{\alpha=-\ell+1/2}^{\ell-1/2} \Phi(x_j^{2\ell}; 2\mathrm{i}(\alpha+\ell)\eta t) &= 2\pi Q_j^{2\ell} + 8\pi\ell\eta(\nu + 2\Sigma_{\mathrm{I}}) \\
&+ \sum_{k=1}^{N/2-2} \left(\sum_{m=1}^{2\ell-1} \Phi(x_j^{2\ell} - x_k^{2\ell}; 2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1} \Phi(x_j^{2\ell} - x_k^{2\ell}; 2\mathrm{i}(m+1)\eta t) \right) \\
&+ \sum_{m=1/2}^{2\ell-3/2} (\Phi(x_j^{2\ell} - x_-^{2\ell-1}; 2\mathrm{i}m\eta t) + \Phi(x_j^{2\ell} - x_-^{2\ell-1}; 2\mathrm{i}(m+1)\eta t)) \\
&+ \sum_{m=1/2}^{2\ell-1/2} (\Phi(x_j^{2\ell} - x_+^{2\ell+1}; 2\mathrm{i}m\eta t) + \Phi(x_j^{2\ell} - x_+^{2\ell+1}; 2\mathrm{i}(m+1)\eta t))
\end{aligned}
\tag{2.3.1}
$$

where $x_\pm^{2\ell\pm1}$ are the centres of the $(2\ell \pm 1)$-strings,

$$
\Sigma_{\mathrm{I}} = (\text{sum of all } x_{j,\alpha}^A) = 2\ell \sum_{j=1}^{N/2-2} x_j^{2\ell} + (2\ell-1)x_-^{2\ell-1} + (2\ell+1)x_+^{2\ell+1}. \tag{2.3.2}
$$

The argument of [38] applied to (2.3.1) implies that there are $N/2$ vacancies for $Q_j^{2\ell}$'s. Thus there remain two vacancies (holes) left unoccupied by centres of 2ℓ-strings.

We renumber the centres of strings as follows.

(i) 2ℓ-strings: x_j, $j = 1, \ldots, N/2$, $j \neq j_1, j_2$, where $Q_{j_1}^{2\ell}$ and $Q_{j_2}^{2\ell}$ correspond to holes. Following the argument in (2.3.1) again, we assume that $x_j > x_{j'}$ if $j > j'$;

(ii) $2\ell - 1$-string: x_-; and

(iii) $2\ell + 1$-string: x_+.

In the thermodynamic limit centres of 2ℓ-strings fill the interval $(-1/2, 1/2)$ continuously with density $\rho_{\mathrm{I}}(x)$ and two holes at $x_1 = \lim x_{j_1}$ and $x_2 = \lim x_{j_2}$. (We abuse indices.) Subtracting (2.3.1) for j from that for $j+1$, we obtain

$$
\begin{aligned}
\sum_{\alpha=-\ell+1/2}^{\ell-1/2} \Phi'(x; 2\mathrm{i}(\alpha+\ell)\eta t) &= -2\pi \left(\rho_{\mathrm{I}}(x) + \frac{1}{N}(\delta(x-x_1) + \delta(x-x_2)) \right) \\
&+ \int_{-1/2}^{1/2} \left(\sum_{m=1}^{2\ell-1} \Phi'(x-y; 2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1} \Phi'(x-y; 2\mathrm{i}(m+1)\eta t) \right) \rho_{\mathrm{I}}(y)\,\mathrm{d}y \\
&+ \frac{1}{N} \sum_{m=1/2}^{2\ell-3/2} (\Phi'(x-x_-; 2\mathrm{i}m\eta t) + \Phi'(x-x_-; 2\mathrm{i}(m+1)\eta t)) \\
&+ \frac{1}{N} \sum_{m=1/2}^{2\ell-1/2} (\Phi'(x-x_+; 2\mathrm{i}m\eta t) + \Phi'(x-x_+; 2\mathrm{i}(m+1)\eta t))
\end{aligned}
\tag{2.3.3}
$$

for large N. The solution of this integral equation for $\rho_{\mathrm{I}}(x)$ is

$$
\rho_{\mathrm{I}}(x) = \rho(x) + \frac{1}{N}(\sigma(x-x_1) + \sigma(x-x_2) + \omega_-(x-x_-) + \omega_+(x-x_+)) \tag{2.3.4}
$$

where $\rho(x)$ is defined above, $\sigma(x)$ and $\omega_\pm(x)$ are solutions of the following integral equations.

Integral equation for $\sigma(x)$:

$$
2\pi\sigma(x) = -2\pi\delta(x) + \int_{-1/2}^{1/2} \left(\sum_{m=1}^{2\ell-1} \Phi'(x-y; 2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1} \Phi'(x-y; 2\mathrm{i}(m+1)\eta t) \right) \sigma(y)\,\mathrm{d}y. \tag{2.3.5}
$$

Integral equation for $\omega_-(x)$:

$$2\pi\omega_-(x) = \int_{-1/2}^{1/2}\left(\sum_{m=1}^{2\ell-1}\Phi'(x-y;2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1}\Phi'(x-y;2\mathrm{i}(m+1)\eta t)\right)\omega_-(y)\,\mathrm{d}y + \sum_{m=1/2}^{2\ell-3/2}(\Phi'(x;2\mathrm{i}m\eta t)+\Phi'(x;2\mathrm{i}(m+1)\eta t)). \tag{2.3.6}$$

Integral equation for $\omega_+(x)$:

$$2\pi\omega_+(x) = \int_{-1/2}^{1/2}\left(\sum_{m=1}^{2\ell-1}\Phi'(x-y;2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1}\Phi'(x-y;2\mathrm{i}(m+1)\eta t)\right)\omega_+(y)\,\mathrm{d}y + \sum_{m=1/2}^{2\ell-1/2}(\Phi'(x;2\mathrm{i}m\eta t)+\Phi'(x;2\mathrm{i}(m+1)\eta t)). \tag{2.3.7}$$

They are easily solved by the Fourier expansion explicitly:

$$\sigma(x) = -\frac{1}{4\ell} - \sum_{n=1}^{\infty}\frac{\sin h\pi nt\,\sin h2\pi n\eta t}{\sin h\pi nt(1-4\ell\eta)\,\sin h4\pi n\ell\eta t\cos h2\pi n\eta t}\cos 2\pi nx \tag{2.3.8}$$

$$\omega_-(x) = -\frac{2\ell-1}{2\ell} - \sum_{n=1}^{\infty}\frac{2\sin h2\pi n(2\ell-1)\eta t}{\sin h4\pi n\ell\eta t}\cos 2\pi nx \tag{2.3.9}$$

$$\omega_+(x) = -1 - \sum_{n=1}^{\infty}\frac{2\sin h\pi nt(1-2(2\ell+1)\eta)}{\sin h\pi nt(1-4\ell\eta)}\cos 2\pi nx. \tag{2.3.10}$$

The product of the Bethe equations (2.1.1) for the $(2\ell-1)$-string $x_- + 2\mathrm{i}\alpha\eta t$, $\alpha = -\ell+1,\dots,\ell-1$, gives the equation:

$$N\sum_{\alpha=-\ell+1}^{\ell+1}\Phi(x_-;2\mathrm{i}(\alpha+\ell)\eta t) = 2\pi Q_-^{2\ell-1} - (2\ell-1)4\pi\eta(\nu+2\Sigma_{\mathrm{I}}) + \sum_{k=1,k\neq j_1,j_2}^{N/2}\sum_{m=1/2}^{2\ell-3/2}(\Phi(x_--x_k;2\mathrm{i}m\eta t)+\Phi(x_--x_k;2\mathrm{i}(m+1)\eta t)) + \sum_{m=1}^{2\ell-1}(\Phi(x_--x_+;2\mathrm{i}m\eta t)+\Phi(x_--x_+;2\mathrm{i}(m+1)\eta t)). \tag{2.3.11}$$

This time there exists only one vacancy for $Q_-^{2\ell-1}$ which determines the branch. We set $Q_-^{2\ell-1}=0$. In the thermodynamic limit equation (2.3.11) gives an integral equation:

$$\sum_{\alpha=-\ell+1}^{\ell+1}\Phi(x_-;2\mathrm{i}(\alpha+\ell)\eta t) = -\frac{1}{N}(2\ell-1)4\pi\eta(\nu+2\Sigma_{\mathrm{I}}) + \int_{-1/2}^{1/2}\sum_{m=1/2}^{2\ell-3/2}(\Phi(x_--y;2\mathrm{i}m\eta t)+\Phi(x_--y;2\mathrm{i}(m+1)\eta t))\rho_{\mathrm{I}}(y)\,\mathrm{d}y + \frac{1}{N}\sum_{m=1}^{2\ell-1}(\Phi(x_--x_+;2\mathrm{i}m\eta t)+\Phi(x_--x_+;2\mathrm{i}(m+1)\eta t)). \tag{2.3.12}$$

This equation reduces to

$$\frac{2\ell}{2\ell-1}\int_{-x_-+x_2}^{x_--x_1}\omega_-(y)\,\mathrm{d}y + x_+ - \frac{x_1+x_2}{2} = (1-4\ell\eta)\Sigma_{\mathrm{I}} - 2\ell\nu\eta \tag{2.3.13}$$

by (2.3.4), (2.2.4), (2.3.8)–(2.3.10) and (C.1).

On the other hand, the product of the Bethe equations (2.2.1) for the $(2\ell+1)$-string $x_+ + 2\,\mathrm{i}\alpha\eta t$, $\alpha = -\ell, \ldots, \ell$, gives the equation:

$$N \sum_{\alpha=-\ell+1}^{\ell} \Phi(x_+; 2\mathrm{i}(\alpha+\ell)\eta t) = 2\pi Q_+^{2\ell+1} - (2\ell+1)4\pi\eta(\nu + 2\Sigma_{\mathrm{I}})$$
$$+ \sum_{k=1, k\neq j_1, j_2}^{N/2} \sum_{m=1/2}^{2\ell-1/2} (\Phi(x_+ - x_k; 2\mathrm{i}m\eta t) + \Phi(x_+ - x_k; 2\mathrm{i}(m+1)\eta t))$$
$$+ \sum_{m=1}^{2\ell-1} (\Phi(x_+ - x_-; 2\mathrm{i}m\eta t) + \Phi(x_+ - x_-; 2\mathrm{i}(m+1)\eta t)). \tag{2.3.14}$$

Again only one vacancy for $Q_+^{2\ell+1}$ which determines the branch exists. We set $Q_+^{2\ell+1} = 0$. The integral equation in the thermodynamic limit given by (2.3.14) is

$$\sum_{\alpha=-\ell+1}^{\ell} \Phi(x_+; 2\mathrm{i}(\alpha+\ell)\eta t) = -\frac{1}{N}(2\ell+1)4\pi\eta(\nu + 2\Sigma_{\mathrm{I}})$$
$$+ \int_{-1/2}^{1/2} \sum_{m=1/2}^{2\ell-1/2} (\Phi(x_+ - y; 2\mathrm{i}m\eta t) + \Phi(x_+ - y; 2\mathrm{i}(m+1)\eta t))\rho_{\mathrm{I}}(y)\,\mathrm{d}y$$
$$+ \frac{1}{N} \sum_{m=1}^{2\ell-1} (\Phi(x_+ - x_-; 2\mathrm{i}m\eta t) + \Phi(x_+ - x_-; 2\mathrm{i}(m+1)\eta t)) \tag{2.3.15}$$

and hence

$$\frac{1}{2} \int_{-x_+ + x_2}^{x_+ - x_1} \omega_+(y)\,\mathrm{d}y + x_+ - \frac{x_1 + x_2}{2} = (1 - 2(2\ell+1)\eta)\Sigma_{\mathrm{I}} - (2\ell+1)\nu\eta \tag{2.3.16}$$

by (2.3.4), (2.2.4), (2.3.8)–(2.3.10) and (C.1).

Let us denote the solution of the Bethe equations for the ground state by x_j^{G}. *Polarization* of the Dirac sea of 2ℓ- strings for excited state I is defined by

$$J(x) := \rho(x) \lim_{N\to\infty} N(x_j - x_j^{\mathrm{G}}) = \lim_{N\to\infty} \frac{x_j - x_j^{\mathrm{G}}}{x_{j+1}^{\mathrm{G}} - x_j^{\mathrm{G}}} \tag{2.3.17}$$

where $x = \lim_{N\to\infty} x_j$. (See [21, 18].) Subtracting (2.3.1) from (2.2.1) and using the integral equation (2.2.3), one can derive the integral equation for $J(x)$:

$$-2\pi J(x) + \int_{-1/2}^{1/2} \left(\sum_{m=1}^{2\ell-1} \Phi'(x-y; 2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1} \Phi'(x-y; 2\mathrm{i}(m+1)\eta t) \right) J(y)\,\mathrm{d}y$$
$$= -8\pi\ell\eta(\nu + 2\Sigma_{\mathrm{I}}) + \sum_{m=1/2}^{2\ell-3/2} (\Phi(x - x_-; 2\mathrm{i}m\eta t) + \Phi(x - x_-; 2\mathrm{i}(m+1)\eta t))$$
$$+ \sum_{m=1/2}^{2\ell-1/2} (\Phi(x - x_+; 2\mathrm{i}m\eta t) + \Phi(x - x_+; 2\mathrm{i}(m+1)\eta t))$$
$$- \sum_{a=1,2} \left(\sum_{m=1}^{2\ell-1} \Phi'(x - x_a; 2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1} \Phi'(x - x_a; 2\mathrm{i}(m+1)\eta t) \right). \tag{2.3.18}$$

Thus the polarization is determined as

$$J(x) = \sum_{n\in\mathbb{Z}} J_n \mathrm{e}^{2\pi\,\mathrm{i}nx} \tag{2.3.19}$$

$$J_0 = \eta(\nu + 2\Sigma_{\mathrm{I}}) - \frac{2\ell-1}{2\ell} x_- - x_+ + \frac{4\ell-1}{4\ell}(x_1 + x_2) \tag{2.3.20}$$

$$J_n = \frac{\sin h2\pi n(2\ell-1)\eta t}{2\pi in \sin h4\pi n\ell\eta t}\left(e^{-2\pi inx_-} - \frac{e^{-2\pi inx_1}+e^{-2\pi inx_2}}{2\cos h2\pi n\eta t}\right) + \frac{\sin h\pi n(1-2(2\ell+1)\eta)t}{2\pi in\sin h\pi n(1-4\ell\eta)t}\left(e^{-2\pi inx_+} - \frac{e^{-2\pi inx_1}+e^{-2\pi inx_2}}{2\cos h2\pi n\eta t}\right). \tag{2.3.21}$$

On the other hand, by the definition of the polarization (2.3.17),

$$2\ell\int_{-1/2}^{1/2} J(x)\,dx = \Sigma_I - (2\ell+1)x_+ - (2\ell-1)x_- + 2\ell(x_1+x_2). \tag{2.3.22}$$

Combining (2.3.20) and (2.3.22), we obtain

$$x_+ - \frac{x_1+x_2}{2} = (1-4\ell\eta)\Sigma_I - 2\ell\nu\eta. \tag{2.3.23}$$

Now we determine $x_\pm$ in terms of x_1, x_2 regarded as free parameters. We have derived three equations connecting x_1, x_2 and $x_\pm$: (2.1.13), (2.3.16) and (2.3.23). From (2.3.13) and (2.3.23) follows

$$\int_{-x_-+x_2}^{x_--x_1} \omega_-(y)\,dy = 0. \tag{2.3.24}$$

Thus $x_- = (x_1+x_2)/2$, since $\omega_-(y) < 0$ because of (2.3.9) and lemma C.2. Equations (2.3.16) and (2.3.23) imply

$$\frac{1}{2}\int_{-x_++x_2}^{x_+-x_1} \omega_+(y)\,dy = -\eta(\nu+2\Sigma_I) \tag{2.3.25}$$

or, equivalently,

$$\int_{-x_++x_2}^{x_+-x_1}\left(\omega_+(y) + \frac{2\eta}{1-4\ell\eta}\right)dy = -\nu\frac{2\eta}{1-4\ell\eta}. \tag{2.3.26}$$

Equation (2.3.26) shows that x_+ is uniquely determined for each ν because of lemma C.2. Constraints (2.3.23), (2.3.26) and (1.4.1) on ν, x_+ and Σ_I are simultaneously satisfied if we put

$$\nu = k\left(\frac{r}{2} - (2\ell+1)r'\right) \qquad \Sigma_I = (2\ell+1)\frac{kr'}{2} \qquad x_+ = \frac{x_1+x_2}{2} + \frac{kr'}{2}$$

where k is an arbitrary integer. Recall that r is assumed to be even. Apparently there are infinitely many solutions, but in fact only two Bethe vectors are independent in this series.

Lemma 2.3.1. For two integers k, k' such that $k \equiv k' \pmod 2$, corresponding Bethe vectors are linearly dependent.

Proof. Let us denote ν and x_+ corresponding to k and k' by $(\nu(k), x_+(k))$ and $(\nu(k'), x_+(k'))$ respectively. Then

$$\begin{aligned}\nu(k)-\nu(k') &= -(2\ell+1)(k-k')r' + \frac{k-k'}{2}r \\ &\equiv -(2\ell+1)(k-k')r' \qquad \pmod r \\ x_+(k) - x_+(k') &= \frac{k-k'}{2}r'.\end{aligned}$$

Shifting x_+ by one means shifting $2\ell+1$ of solutions of Bethe equations by one. The lemma is proved by lemma 1.3.2. □

If we take $k = 0$, then $\nu = 0$, $\Sigma_{\mathrm{I}} = 0$ and $x_+ = (x_1 + x_2)/2$. We call the Bethe vector corresponding to this configuration *excited state* I_0.

Since r and r' are coprime, there exists an integer k such that $kr' \equiv 1 \pmod r$. Shifting x_+ by an integer, we may assume that $x_+ = (x_1 + x_2)/2 + 1/2$ with a suitable ν. (See lemma 1.3.2.) We call this Bethe vector *excited state* I_1.

It seems that there are no other solutions for x_+ and ν, since the integrality conditions ($\nu \in \mathbb{Z}$ and theorem 1.4.1) are very strong.

2.3.2. *Excited state II.* Now we consider the excited state II. As in the case of the excited state I, multiplying the Bethe equations (2.2.1) for a 2ℓ-string $x_i = x_{j,\alpha}^{2\ell,+}$, $\alpha = -\ell + 1/2, \ldots, \ell - 1/2$ with the centre $x_j^{2\ell,+}$, and taking the logarithm, we obtain

$$\begin{aligned}
N \sum_{\alpha=-\ell+1/2}^{\ell-1/2} \Phi(x_j^{2\ell,+}; 2\mathrm{i}(\alpha+\ell)\eta t) &= 2\pi Q_j^{2\ell} + 8\pi\ell\eta(\nu + 2\Sigma_{\mathrm{II}}) \\
&\quad + \sum_{k=1}^{N/2-1} \left(\sum_{m=1}^{2\ell-1} \Phi(x_j^{2\ell,+} - x_k^{2\ell,+}; 2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1} \Phi(x_j^{2\ell,+} - x_k^{2\ell,+}; 2\mathrm{i}(m+1)\eta t) \right) \\
&\quad + \sum_{m=1/2}^{2\ell-3/2} (\Phi(x_j^{2\ell,+} - x_-^{2\ell-1,+}; 2\mathrm{i}m\eta t) + \Phi(x_j^{2\ell,+} - x_-^{2\ell-1,+}; 2\mathrm{i}(m+1)\eta t)) \\
&\quad + \Psi(x_j^{2\ell,+} - x_0; 2\mathrm{i}(2\ell+1)\eta t) + \Psi(x_j^{2\ell,+} - x_0; 2\mathrm{i}(2\ell-1)\eta t)
\end{aligned} \tag{2.3.27}$$

where $x_-^{2\ell-1,+}$ is the centre of the $(2\ell-1)$-string, $\{x_0 + it/2\}$ ($x_0 \in \mathbb{R}$) is the one-string with parity $-$,

$$\Sigma_{\mathrm{II}} \doteq (\text{sum of all } x_{j,\alpha}^A) - \frac{\mathrm{i}t}{2} = 2\ell \sum_{j=1}^{N/2-1} x_j^{2\ell,+} + (2\ell-1)x^{2\ell-1,+} + x_0^{1,-} \tag{2.3.28}$$

and function $\Psi(x; \mathrm{i}\mu t)$ is defined by

$$\Psi(x; \mathrm{i}\mu t) = \frac{1}{\mathrm{i}} \log \frac{\theta_{01}(x + \mathrm{i}\mu t; \mathrm{i}t)}{\theta_{01}(x - \mathrm{i}\mu t; \mathrm{i}t)}. \tag{2.3.29}$$

By the same argument as for the excited state I, there are $N/2 + 1$ vacancies for $Q_j^{2\ell}$'s. Thus there are again two holes of centres of 2ℓ-strings.

We renumber the centres of strings as follows.

(i) 2ℓ-strings: x_j, $j = 1, \ldots, N/2+1$, $j \neq j_1, j_2$, where $Q_{j_1}^{2\ell}$ and $Q_{j_2}^{2\ell}$ correspond to holes. Following the argument in (2.3.1) again, we assume that $x_j > x_{j'}$ if $j > j'$. The string with its centre at $x_{N/2+1}$ will be placed at the zone boundary $x = 1/2$ in the thermodynamic limit.

(ii) $(2\ell-1)$-string: x_-.

(iii) 1-string with parity $-$: $x_0 + \frac{1}{2}it$.

As in the previous case, we obtain an integral equation for the density of centres of 2ℓ-strings, $\rho_{\mathrm{II}}(x)$, on the interval $(-1/2, 1/2)$

$$\begin{aligned}
\sum_{\alpha=-\ell+1/2}^{\ell-1/2} \Phi'(x_j^{2\ell}; 2\mathrm{i}(\alpha+\ell)\eta t) &= -2\pi \left(\rho_{\mathrm{II}}(x) + \frac{1}{N}(\delta(x - x_1) + \delta(x - x_2)) \right) \\
&\quad + \int_{-1/2}^{1/2} \left(\sum_{m=1}^{2\ell-1} \Phi'(x - y; 2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1} \Phi'(x - y; 2\mathrm{i}(m+1)\eta t) \right) \rho_{\mathrm{II}}(y)\,\mathrm{d}y
\end{aligned}$$

$$+\frac{1}{N}\sum_{m=1/2}^{2\ell-3/2}(\Phi'(x-x_-;2\mathrm{i}m\eta t)+\Phi'(x-x_-;2\mathrm{i}(m+1)\eta t))$$
$$+\frac{1}{N}\Psi'(x-x_0;2\mathrm{i}(2\ell+1)\eta t)+\Psi'(x-x_0;2\mathrm{i}(2\ell-1)\eta t) \tag{2.3.30}$$

for large N. Its solution is

$$\rho_{\mathrm{II}}(x)=\rho(x)+\frac{1}{N}(\sigma(x-x_1)+\sigma(x-x_2)+\omega_-(x-x_-)+\omega_0(x-x_0)) \tag{2.3.31}$$

where $\rho(x)$ ((2.2.3), (2.2.4)), $\sigma(x)$ (2.3.5), (2.3.8)), $\omega_-(x)$ ((2.3.6), (2.3.9)) are as defined before, and $\omega_0(x)$ is a solution of the following integral equation:

$$2\pi\omega_0(x)=\int_{-1/2}^{1/2}\left(\sum_{m=1}^{2\ell-1}\Phi'(x-y;2\mathrm{i}m\eta t)+\sum_{m=0}^{2\ell-1}\Phi'(x-y;2\mathrm{i}(m+1)\eta t)\right)\omega_0(y)\,\mathrm{d}y$$
$$+\Psi'(x;\mathrm{i}(2\ell+1)\eta t)+\Psi'(x;\mathrm{i}(2\ell-1)\eta t). \tag{2.3.32}$$

Explicitly, $\omega_0(x)$ is

$$\omega_0(x)=\sum_{n=1}^{\infty}\frac{2\sin h2\pi n\eta t}{\sin h\pi n(1-4\ell\eta)t}\cos 2\pi nx. \tag{2.3.33}$$

The Bethe equations (2.2.1) for the $(2\ell-1)$-string gives the equation:

$$N\sum_{\alpha=-\ell+1}^{\ell+1}\Phi(x_-;2\mathrm{i}(\alpha+\ell)\eta t)=2\pi Q_-^{2\ell-1}-(2\ell-1)4\pi\eta(\nu+2\Sigma_{\mathrm{II}})$$
$$+\sum_{k=1,\neq j_1,j_2}^{N/2+1}\sum_{m=1/2}^{2\ell-3/2}(\Phi(x_--x_k;2\mathrm{i}m\eta t)+\Phi(x_--x_k;2\mathrm{i}(m+1)\eta t))$$
$$+\Psi(x_--x_0;2\mathrm{i}\ell\eta t)+\Psi(x_--x_0;2\mathrm{i}(\ell-1)\eta t). \tag{2.3.34}$$

We set $Q_-^{2\ell-1}=0$, since there is only one vacancy. The corresponding integral equation in the thermodynamic limit is

$$\sum_{\alpha=-\ell+1}^{\ell+1}\Phi(x_-;2\mathrm{i}(\alpha+\ell)\eta t)=-\frac{1}{N}(2\ell-1)4\pi\eta(\nu+2\Sigma_{\mathrm{II}})$$
$$+\int_{-1/2}^{1/2}\sum_{m=1/2}^{2\ell-3/2}(\Phi(x_--y;2\mathrm{i}m\eta t)+\Phi(x_--y;2\mathrm{i}(m+1)\eta t))\rho_{\mathrm{II}}(y)\,\mathrm{d}y$$
$$+\frac{1}{N}(\Psi(x_--x_0;2\mathrm{i}\ell\eta t)+\Psi(x_--x_0;2\mathrm{i}(\ell-1)\eta t)). \tag{2.3.35}$$

This equation reduces to

$$\frac{\ell}{2\ell-1}\int_{-x_-+x_2}^{x_--x_1}\omega_-(y)\,\mathrm{d}y+x_0-\frac{x_1+x_2}{2}=(1-4\ell\eta)\Sigma_{\mathrm{II}}-2\ell\nu\eta \tag{2.3.36}$$

as before.

The Bethe equations (2.2.1) for the 1-string with parity $-$ gives the equation:

$$N\Psi(x_0;2\mathrm{i}\ell\eta t)=2\pi Q_0^{1-}-4\pi\eta(\nu+2\Sigma_{\mathrm{II}})$$
$$+\sum_{k=1,\neq j_1,j_2}^{N/2}(\Psi(x_0-x_k;\mathrm{i}(2\ell+1)\eta t)+\Psi(x_0-x_k;\mathrm{i}(2\ell-1)\eta t))$$
$$+\Psi(x_0-x_-;2\mathrm{i}\ell\eta t)+\Psi(x_0-x_-;2\mathrm{i}(\ell-1)\eta t). \tag{2.3.37}$$

Again we choose the branch $Q_0^{1-} = 0$. The integral equation is

$$\Psi(x_0; 2\mathrm{i}\ell\eta t) = -\frac{1}{N}4\pi\eta(\nu + 2\Sigma_{\mathrm{II}}) + \int_{-1/2}^{1/2} (\Psi(x_0 - y; \mathrm{i}(2\ell+1)\eta t) + \Psi(x_0 - y; \mathrm{i}(2\ell-1)\eta t))\rho_{\mathrm{II}}(y)\,\mathrm{d}y + \frac{1}{N}(\Psi(x_0 - x_-; 2\mathrm{i}\ell\eta t) + \Psi(x_0 - x_-; 2\mathrm{i}(\ell-1)\eta t)). \tag{2.3.38}$$

This is rewritten as

$$\int_{-x_0+x_2}^{x_0-x_1} \omega_0(y)\,\mathrm{d}y = -2\eta(2\Sigma_{\mathrm{II}} + \nu). \tag{2.3.39}$$

Let us compute the polarization $J(x)$ of this state. Subtracting (2.3.27) from (2.2.1) and using the integral equation (2.2.3), we can derive the integral equation for $J(x)$:

$$\begin{aligned} &-2\pi J(x) + \int_{-1/2}^{1/2}\left(\sum_{m=1}^{2\ell-1}\Phi'(x-y; 2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1}\Phi'(x-y; 2\mathrm{i}(m+1)\eta t)\right)J(y)\,\mathrm{d}y \\ &\quad = 8\pi\ell\eta(\nu + 2\Sigma_{\mathrm{II}}) + \sum_{m=1/2}^{2\ell-3/2}(\Phi(x-x_-; 2\mathrm{i}m\eta t) + \Phi(x-x_-; 2\mathrm{i}(m+1)\eta t)) \\ &\qquad + \Psi(x-x_0; \mathrm{i}(2\ell+1)\eta t) + \Psi(x-x_0; \mathrm{i}(2\ell-1)\eta t) \\ &\qquad + \sum_{m=1}^{2\ell-1}\Phi'(x-\tfrac{1}{2}; 2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1}\Phi'(x-\tfrac{1}{2}; 2\mathrm{i}(m+1)\eta t) \\ &\qquad - \sum_{a=1,2}\left(\sum_{m=1}^{2\ell-1}\Phi'(x-x_a; 2\mathrm{i}m\eta t) + \sum_{m=0}^{2\ell-1}\Phi'(x-x_a; 2\mathrm{i}(m+1)\eta t)\right). \end{aligned} \tag{2.3.40}$$

Hence the polarization is

$$J(x) = \sum_{n\in\mathbb{Z}} J_n \mathrm{e}^{2\pi\mathrm{i}nx} \tag{2.3.41}$$

$$J_0 = -\frac{x}{4\ell} + \eta(\nu + 2\Sigma_{\mathrm{II}}) - \frac{1}{2} - \frac{2\ell-1}{2\ell}x_- + \frac{4\ell-1}{4\ell}(x_1 + x_2) \tag{2.3.42}$$

$$\begin{aligned} J_n &= \frac{\sinh 2\pi n(2\ell-1)\eta t}{\sinh 4\pi n\ell\eta t} \\ &\quad\times\left(\mathrm{e}^{-2\pi\mathrm{i}nx_-} - \frac{-\mathrm{e}^{-\pi\mathrm{i}n} + \mathrm{e}^{-2\pi\mathrm{i}nx_1} + \mathrm{e}^{-2\pi\mathrm{i}nx_2}}{2\cosh 2\pi n\eta t} - \frac{\mathrm{e}^{-2\pi\mathrm{i}nx_0}}{\sinh \pi nt(1-4\ell\eta)}\right) \\ &\quad + \frac{\sinh \pi n(1-2(2\ell+1)\eta)t}{2\cosh 2\pi n\eta t\,\sinh \pi n(1-4\ell\eta)t}(-\mathrm{e}^{-\pi\mathrm{i}n} + \mathrm{e}^{-2\pi\mathrm{i}nx_1} + \mathrm{e}^{-2\pi\mathrm{i}nx_2}). \end{aligned} \tag{2.3.43}$$

On the other hand, by the definition of the polarization (2.3.17),

$$2\ell\int_{-1/2}^{1/2} J(x)\,\mathrm{d}x = \Sigma_{\mathrm{II}} - (2\ell-1)x_- - x_0 - \ell + 2\ell(x_1 + x_2). \tag{2.3.44}$$

It follows from (2.3.44) and (2.3.42) that

$$x_0 - \frac{x_1 + x_2}{2} = (1 - 4\ell\eta)\Sigma_{\mathrm{II}} - 2\ell\nu\eta. \tag{2.3.45}$$

From (2.3.36) and (2.3.45) follows the same equation as (2.3.24) and thus $x_- = (x_1 + x_2)/2$ as in the case of the excited state I. Equations (2.3.39) and (2.3.45) imply

$$\int_{-x_0+x_2}^{x_0-x_1}\left(\omega_0(y) + \frac{2\eta}{1-4\ell\eta}\right)\mathrm{d}y = -\nu\frac{2\eta}{1-4\ell\eta}. \tag{2.3.46}$$

This equation is uniquely solved because of (2.3.33) and lemma C.2: $x_0 = (x_1+x_2)/2-\nu/2$. Lemma 1.3.2 tells that only two cases give independent Bethe vectors: ($\nu = 0$, $x_0 = (x_1+x_2)/2$) which we call excited state II_0 and ($\nu = 1$, $x_0 = (x_1+x_2)/2+\frac{1}{2}$) which we call excited state II_1.

Table 1. Two-particle excited states.

Excited states	2ℓ-strings parity +	$2\ell+1$-string parity +	$2\ell-1$- string parity +	1-string parity −
I_0	density ρ_{I} holes x_1, x_2	$x_+ = (x_1+x_2)/2$	$x_- = (x_1+x_2)/2$	
I_1	density ρ_{I} holes x_1, x_2	$x_+ = (x_1+x_2+1)/2$	$x_- = (x_1+x_2)/2$	
II_0	density ρ_{II} holes x_1, x_2		$x_- = (x_1+x_2)/2$	$x_0 = (x_1+x_2)/2$
II_1	density ρ_{II} holes x_1, x_2		$x_- = (x_1+x_2)/2$	$x_0 = (x_1+x_2+1)/2$

S-matrix. Above we found four excited states with two free parameters x_1, x_2: I_0, I_1, II_0, II_1. In the rational limit, $t \to \infty$, $\eta \to 0$, ηt fixed, the string configuration of I_0 becomes that of the singlet state of the corresponding spin chain, whereas the configurations I_1, II_0, II_1 seem to approach that of the triplet states, since the one-string with parity − and the string with abscissa $x_+ = (x_1+x_2+1)/2$ goes beyond the sight. (Recall that real abscissas of strings are rescaled so that they fill the whole real line in the limit.) Hence one might expect that these four Bethe vectors give four-dimensional space of two physical particle states (spin waves) of the corresponding spin chain. In fact for the eight-vertex model

$$\log T(x)|_{\text{excited state}} - \log T(x)|_{\text{ground state}} = \log\tau(x-\mathrm{i}\eta t-x_1)+\log\tau(x-\mathrm{i}\eta t-x_2) \tag{2.3.47}$$

where (excited state) means any one of the excited states I_0, I_1, II_0, II_1 and

$$\log\tau(x) := -\frac{\pi\mathrm{i}}{2} - \pi\,\mathrm{i}x - \mathrm{i}\sum_{n=1}^{\infty}\frac{\sin 2\pi nx}{n\cos h2\pi n\eta t} \tag{2.3.48}$$

(see [18] for details of calculations). This means that all conserved quantities such as momentum $P(x)$ or energy over the ground states are split into two terms:

$$P(x) = -\pi x - \sum_{n=1}^{\infty}\frac{\sin 2\pi nx}{n\cos h2\pi n\eta t}$$

and thus we can regard these excited states as two-particle states of the XYZ spin chain.

For higher spin cases we have not yet computed fused transfer matrix which corresponds to the spin chain with local interaction. From the result of the rational and trigonomeric models [36], [20], we conjecture that the momentum and the energy of physical particles do not depend on the spin ℓ. Based on this conjecture, we calculate the S-matrix of two physical particles.

As is discussed in [11], the S-matrix of physical particles (spin waves) could depend on the way of calculation for the case of higher spin. In order to make our standpoint clear, let us recall the calculation of eigenvalues of the S-matrix in more detail, following [21] and section 9 of [25] (cf also [10], [38]): T e appearance of an integer $Q_j^{2\ell}$ in (2.3.1) and

(2.3.27) can be interpreted as a consequence of the periodic boundary condition which we imposed on the lattice. Namely, if we move a physical particle around the whole chain, the total phase shift of the wavefunction accumulated should be an integer multiple of $2\pi\mathrm{i}$. The main contribution comes from the momentum P of the particle as $\mathrm{i}PN$, the *free phase*:

$$\mathrm{i}PN = -\mathrm{i}N\pi x - \mathrm{i}N\sum_{n=1}^{\infty}\frac{\sin 2\pi nx}{n\cos h2\pi n\eta t}.$$

Note that the right-hand side of the above equation is eventually expressed as the logarithm of (the right-hand side)/(the left-hand side) of the Bethe equation for the ground state (2.2.1):

$$\begin{aligned}
&-\mathrm{i}N\sum_{\alpha=-\ell+1/2}^{\ell-1/2}\Phi(x;2\mathrm{i}(\alpha+\ell)\eta t)\\
&\qquad+\mathrm{i}\sum_{k=1}^{N/2}\left(\sum_{m=1}^{2\ell-1}\Phi(x-x_k;2\mathrm{i}m\eta t)+\sum_{m=0}^{2\ell-1}\Phi(x-x_k;2\mathrm{i}(m+1)\eta t)\right)\\
&\qquad\xrightarrow{N\to\infty}-\mathrm{i}N\sum_{\alpha=-\ell+1/2}^{\ell-1/2}\Phi(x;2\mathrm{i}(\alpha+\ell)\eta t)\\
&\qquad+\mathrm{i}N\int_{-1/2}^{1/2}\left(\sum_{m=1}^{2\ell-1}\Phi(x-y;2\mathrm{i}m\eta t)+\sum_{m=0}^{2\ell-1}\Phi(x-y;2\mathrm{i}(m+1)\eta t)\right)\rho(y)\,\mathrm{d}y\\
&\quad=\mathrm{i}PN.
\end{aligned}\tag{2.3.49}$$

Hence the ground state can be interpreted as a Dirac sea of non-interacting particles, since the momenta of particles are integer multiples of $2\pi/N$ because of (2.2.1) and the above equation.

For the excited state, however, the phase shift comes not only from this free phase but also from the interaction between physical particles. Because of the periodic boundary condition which fixes the total phase shift to an integer multiple of $2\pi\mathrm{i}$, this means that calculating the scattering phase shift of a physical particle is equivalent to the calculation of $\mathrm{O}(1/N)$ shift of the momentum. In other words, the S-matrix of physical particles can be calculated by splitting the total phase shift, an integer multiple of $2\pi\mathrm{i}$, into the free phase $\mathrm{i}PN$ of order $\mathrm{O}(N)$ and the scattering phase of order $\mathrm{O}(1)$.

We consider the excited state I first. The total phase shift for the physical particle with rapidity x_1 can be read off from (2.3.1) as follows:

$$\begin{aligned}
&-N\sum_{\alpha=-\ell+1/2}^{\ell-1/2}\Phi(x_1;2\mathrm{i}(\alpha+\ell)\eta t)+8\pi\ell\eta(\nu+2\Sigma_{\mathrm{I}})\\
&\qquad+\sum_{k=1}^{N/2-2}\left(\sum_{m=1}^{2\ell-1}\Phi(x_1-x_k^{2\ell};2\mathrm{i}m\eta t)+\sum_{m=0}^{2\ell-1}\Phi(x_1-x_k^{2\ell};2\mathrm{i}(m+1)\eta t)\right)\\
&\qquad+\sum_{m=1/2}^{2\ell-3/2}(\Phi(x_1-x_-^{2\ell-1};2\mathrm{i}m\eta t)+\Phi(x_1-x_-^{2\ell-1};2\mathrm{i}(m+1)\eta t))\\
&\qquad+\sum_{m=1/2}^{2\ell-1/2}(\Phi(x_1-x_+^{2\ell+1};2\mathrm{i}m\eta t)+\Phi(x_1-x_+^{2\ell+1};2\mathrm{i}(m+1)\eta t))
\end{aligned}\tag{2.3.50}$$

which is equal to an integer multiple of $2\pi\mathrm{i}$ because of (2.3.1). Subtracting the free phase contribution $\mathrm{i}PN$ (2.3.49) from the total phase (2.3.50), and taking the limit $N\to\infty$, we obtain the remainder of order $\mathrm{O}(1)$ and thus we can interpret it as the scattering phase shift from the above argument. An explicit expression for the eigenvalue of the S-matrix for the excited state I is as follows:

$$
\begin{aligned}
\mathrm{i}\log(\pm S_{\mathrm{I}}(x_{\mathrm{I}}-x_2)) = & -8\pi\ell\eta(\nu+2\Sigma_{\mathrm{I}}) \\
& +N\int_{-1/2}^{1/2}\left(\sum_{m=1}^{2\ell-1}\Phi(x_{\mathrm{I}}-y;2\mathrm{i}m\eta t)\right. \\
& \left.+\sum_{m=0}^{2\ell-1}\Phi(x_{\mathrm{I}}-y;2\mathrm{i}(m+1)\eta t)\right)(\rho_{\mathrm{I}}(y)-\rho(y))\,\mathrm{d}y \\
& +\sum_{m=1/2}^{2\ell-3/2}(\Phi(x_{\mathrm{I}}-x_-;2\mathrm{i}m\eta t)+\Phi(x_{\mathrm{I}}-x_-;2\mathrm{i}(m+1)\eta t)) \\
& +\sum_{m=1/2}^{2\ell-1/2}(\Phi(x_{\mathrm{I}}-x_+;2\mathrm{i}m\eta t)+\Phi(x_{\mathrm{I}}-x_+;2\mathrm{i}(m+1)\eta t)).
\end{aligned}
\tag{2.3.51}
$$

The term $-8\pi\ell\eta\nu$ can be interpreted as an effect from the background or boundary, while the rest of the right-hand side comes from interaction of pseudo-particles. The ambiguity of sign comes from normalizations of asymptotic states. The right-hand side is computed by integrating (2.3.5)–(2.3.7). The result is

$$
\begin{aligned}
\mathrm{i}\log(\pm S_{\mathrm{I}}(x)) = & \sum_{n=1}^{\infty}\left(\frac{\sin h\pi nt(1-4\ell\eta-2\eta)}{n\sin h\pi nt(1-4\ell\eta)\,\cos h2\pi n\eta t}\right. \\
& \left.+\frac{\sin h\pi nt(4\ell\eta-2\eta)}{n\sin h4\pi n\ell\eta t\,\cos h2\pi n\eta t}\right)\sin 2\pi nx \\
& +\sum_{n=1}^{\infty}\frac{2\sin h\pi nt(4\ell\eta-2\eta)}{\sin h4\pi n\ell\eta t}\sin\pi nx \\
& +\sum_{n=1}^{\infty}\frac{\sin h\pi nt(2\eta-(1-4\ell\eta))}{\sin h\pi nt(1-4\ell\eta)}\sin\pi n(x-\varepsilon)
\end{aligned}
\tag{2.3.52}
$$

where ε is 0 or 1 for the excited state I_0 or I_1, respectively. The first term of the right-hand side come from holes ($\sigma(x-x_2)$ in $\rho_{\mathrm{I}}(x)-\rho(x)$ of (2.3.51)), the second term from the $(2\ell-1)$-string ($\omega_-(x-x_-)$) and the last term from the $(2\ell+1)$-string ($\omega_+(x-x_+)$).

Computation for the excited state II is the same. The result is

$$
\begin{aligned}
\mathrm{i}\log(\pm S_{\mathrm{II}}(x)) = & \sum_{n=1}^{\infty}\left(\frac{\sin h\pi nt(1-4\ell\eta-2\eta)}{n\sin h\pi nt(1-4\ell\eta)\,\cos h2\pi n\eta t}\right. \\
& \left.+\frac{\sin h\pi nt(4\ell\eta-2\eta)}{n\sin h4\pi n\ell\eta t\,\cos h2\pi n\eta t}\right)\sin 2\pi nx \\
& +\sum_{n=1}^{\infty}\frac{2\sin h\pi nt(4\ell\eta-2\eta)}{\sin h4\pi n\ell\eta t}\sin\pi nx \\
& +\pi+\pi x+\sum_{n=1}^{\infty}\frac{\sin h2\pi n\eta t}{\sin h\pi nt(1-4\ell\eta)}\sin\pi n(x-\varepsilon)
\end{aligned}
\tag{2.3.53}
$$

where ε is 0 or 1 for the excited states II_0 or II_1, respectively. As in the case of the excited state I, the first term of the right-hand side come from holes, the second term from the $(2\ell-1)$-string and the last term from the one-string with parity $-$ ($\omega_0(x-x_0)$).

We fix the signs left undetermined so that the above S-matrix is the permutation matrix in the non-interacting limit $x=0$ and the excited state I_0 reduces to a singlet while the other three states form a triplet as in the rational and trigonometric cases. Then,

$$
\begin{aligned}
S(x)|_{\text{excited state } I_0} &= S_0(x)\frac{\theta_{11}(\frac{x}{2} - \mathrm{i}t\eta; \mathrm{i}t(1-4\ell\eta))}{\theta_{11}(\frac{x}{2} + \mathrm{i}t\eta; \mathrm{i}t(1-4\ell\eta))} \\
S(x)|_{\text{excited state } I_1} &= S_0(x)\frac{\theta_{10}(\frac{x}{2} - \mathrm{i}t\eta; \mathrm{i}t(1-4\ell\eta))}{\theta_{10}(\frac{x}{2} + \mathrm{i}t\eta; \mathrm{i}t(1-4\ell\eta))} \\
S(x)|_{\text{excited state } II_0} &= S_0(x)\frac{\theta_{01}(\frac{x}{2} - \mathrm{i}t\eta; \mathrm{i}t(1-4\ell\eta))}{\theta_{01}(\frac{x}{2} + \mathrm{i}t\eta; \mathrm{i}t(1-4\ell\eta))} \\
S(x)|_{\text{excited state } II_1} &= S_0(x)\frac{\theta_{00}(\frac{x}{2} - \mathrm{i}t\eta; \mathrm{i}t(1-4\ell\eta))}{\theta_{00}(\frac{x}{2} + \mathrm{i}t\eta; \mathrm{i}t(1-4\ell\eta))}
\end{aligned}
\tag{2.3.54}
$$

where

$$
S_0(x) = \mathrm{e}^{-2\pi\mathrm{i}x}\frac{\theta_{11}\left(\frac{x}{2} - \mathrm{i}t\eta; 4\ell\mathrm{i}t\eta\right)}{\theta_{11}\left(\frac{x}{2} + \mathrm{i}t\eta; 4\ell\mathrm{i}t\eta\right)}\mathbb{S}(x; 1-4\ell\eta)\mathbb{S}(x; 4\ell\eta)
\tag{2.3.55}
$$

and function $\mathbb{S}(x;\mu)$ is defined by

$$
\begin{aligned}
\mathbb{S}(\mathrm{i}\lambda t; \mu) &= \exp\left(\sum_{n=1}^{\infty} \frac{\sin h\pi nt(\mu - 2\eta)}{n \sin h\pi nt\mu \, \cos h2\pi nt\eta} \sin 2\pi n \,\mathrm{i}\lambda t\right) \\
&= \frac{(q^4p^\lambda; p^\mu, q^4)_\infty (p^{\lambda+\mu}; p^\mu, q^4)_\infty (q^2p^{-\lambda}; p^\mu, q^4)_\infty (q^2p^{-\lambda+\mu}; p^\mu, q^4)_\infty}{(q^4p^{-\lambda}; p^\mu, q^4)_\infty (p^{-\lambda+\mu}; p^\mu, q^4)_\infty (q^2p^{\lambda}; p^\mu, q^4)_\infty (q^2p^{\lambda+\mu}; p^\mu, q^4)_\infty} \\
&= \frac{\Gamma_{q^4}\left(\frac{1}{2} + \frac{\lambda}{4\eta}\right)\Gamma_{q^4}\left(1 - \frac{\lambda}{4\eta}\right)}{\Gamma_{q^4}\left(\frac{1}{2} - \frac{\lambda}{4\eta}\right)\Gamma_{q^4}\left(1 + \frac{\lambda}{4\eta}\right)} \\
&\quad \times \prod_{k=1}^{\infty} \frac{\Gamma_{q^4}\left(\frac{1}{2} + \frac{\lambda+k\mu}{4\eta}\right)^2 \Gamma_{q^4}\left(1 + \frac{-\lambda+k\mu}{4\eta}\right)\Gamma_{q^4}\left(\frac{-\lambda+k\mu}{4\eta}\right)}{\Gamma_{q^4}\left(\frac{1}{2} + \frac{-\lambda+k\mu}{4\eta}\right)^2 \Gamma_{q^4}\left(1 + \frac{\lambda+k\mu}{4\eta}\right)\Gamma_{q^4}\left(\frac{\lambda+k\mu}{4\eta}\right)}
\end{aligned}
\tag{2.3.56}
$$

where $p = \mathrm{e}^{-2\pi t}$, $q = \mathrm{e}^{-2\pi\eta t} = p^\eta$. (See appendix C for definitions of notations. The last equality is due to (C.7).) This $\mathbb{S}$ factor was found by Freund and Zabrodin [13].

Comparing (2.3.54) with (A.12), we come to the following conclusion (cf [12]):

$$
S(x) \propto R(\lambda; \mathrm{i}t(1-4\ell\eta)).
\tag{2.3.57}
$$

3. Comments and discussions

In this paper we have studied the eigenvectors of transfer matrix of higher spin generalizations of the eight-vertex model by means of the Bethe ansatz. Apparently the Bethe ansatz for these models seems to be less powerful compared to Bethe ansatz for the XXX model, the XXZ model and their higher spin generalizations, since the number of quasi-particles (B operators) are restricted to $N\ell$. However, we have a discrete parameter ν instead. In chapter 2, varying this parameter, we restored all two-particle states which would degenerate to a singlet and a triplet in the limit, $\eta \to 0$, $t \to \infty$. Therefore we can expect that the Bethe ansatz for our case gives as many eigenvectors as that for the rational and trigonometric cases.

Developments of the theory of quantum affine algebras in the last decade provided algebraic tools such as vertex operators and a crystal basis for investigation of the models associated with trigonometric R-matrices. This kind of algebraic method is still hard to apply to the models associated with elliptic R-matrices because of the lack of knowledge of 'elliptic affine algebras' which should be an affinization of the Sklyanin algebra in an appropriate sense. Foda *et al* [12] proposed a candidate for this algebra. In their formulation a relation of the type $RLL = LLR^*$ plays an important role, where R^* is essentially the S-matrix of two-particle states of the XYZ model. Their argument was based on Smirnov's conjecture, which is supported by the result of the present paper.

The algebra which Foda *et al* propose is considered to be a symmetry of the XYZ spin chain in a thermodynamic limit. We can also expect that their algebra is also a symmetry algebra for higher spin models which we considered in this paper. On the other hand, it is still not known whether finite size models could have a symmetry of the Sklyanin algebra, since a reasonable coproduct for the Sklyanin algebra has not yet been found.

We considered only such excitations with finite number of holes and finite number of strings which have length A, $A \neq 2\ell$, even in the thermodynamic limit. This is because we wanted to determine the two-particle S-matrix. When we want to calculate thermodynamic quantities such as the entropy or specific heat of the model, string configurations with a non-zero hole density and non-zero densities of A-strings ($A \neq 2\ell$) are essential. (See [39, 33, 20].) The extensive thermodynamics of the XXX, XXZ models and their generalization to higher spin cases has quite interesting features [29], and are also related to dilogarithm identities [2], [20]. Further study of the thermodynamic Bethe ansatz for higher spin generalizations of the eight-vertex model could give the deformation of the above features in rational and trigonometric models.

Acknowledgments

The author expresses his gratitude to E K Sklyanin and J Suzuki for valuable advice and discussions. He also thanks to K Hasegawa, K Iohara, M Jimbo, A N Kirillov, V Korepin, A Kuniba, T Miwa, M Noumi, N Yu Reshetikhin, N Slavnov, F Smirnov, Y Yamada and A Zabrodin for their comments, suggestions and interest.

This work is partly supported by the Grant-in-Aid for Scientific Research on Priority Areas 231, the Ministry of Education, Science and Culture.

Appendix A. Review of the Sklyanin algebra

In this appendix we recall several facts on the Sklyanin algebra and its representations from [30] and [31]. We use the notation in [26] for theta functions:

$$\theta_{ab}(z;\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi\,\mathrm{i}\left(\frac{a}{2}+n\right)^2\tau + 2\pi\,\mathrm{i}\left(\frac{a}{2}+n\right)\left(\frac{b}{2}+z\right)\right) \tag{A.1}$$

where τ is a complex number such that $\mathrm{Im}(\tau) > 0$. We denote $t = \mathrm{i}/\tau$. The Pauli matrices are defined as usual:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.2}$$

The *Sklyanin algebra*, $U_{\tau,\eta}(sl(2))$ is generated by four generators S^0, S^1, S^2, S^3, satisfying the following relations:

$$R_{12}(\lambda-\mu)L_{01}(\lambda)L_{02}(\mu) = L_{02}(\mu)L_{01}(\lambda)R_{12}(\lambda-\mu). \tag{A.3}$$

Here λ, μ are complex parameters, the L *operator*, $L(\lambda)$, is defined by

$$L(\lambda) = \sum_{a=0}^{3} W_a^L(\lambda) S^a \otimes \sigma^a$$

$$W_0^L(\lambda) = \frac{\theta_{11}(\lambda;\tau)}{\theta_{11}(\eta;\tau)} \qquad W_1^L(\lambda) = \frac{\theta_{10}(\lambda;\tau)}{\theta_{10}(\eta;\tau)}$$
$$W_2^L(\lambda) = \frac{\theta_{00}(\lambda;\tau)}{\theta_{00}(\eta;\tau)} \qquad W_3^L(\lambda) = \frac{\theta_{01}(\lambda;\tau)}{\theta_{01}(\eta;\tau)} \tag{A.4}$$

$R(\lambda) = R(\lambda; \mathrm{i}t)$ is *Baxter's R-matrix* defined by

$$R(\lambda) = \sum_{a=0}^{3} W_a^R(\lambda)\sigma^a \otimes \sigma^a \qquad W_a^R(\lambda) := W_a^L(\lambda+\eta) \tag{A.5}$$

and indices $\{0, 1, 2\}$ denote the spaces on which operators act non-trivially: for example,

$$R_{12}(\lambda) = \sum_{a=0}^{3} W_a^R(\lambda) 1 \otimes \sigma^a \otimes \sigma^a \qquad L_{01}(\lambda) = \sum_{a=0}^{3} W_a^L(\lambda) S^a \otimes \sigma^a \otimes 1.$$

The above relation (A.3) contains λ and μ as parameters, but the commutation relations among S^a ($a = 0, \ldots, 3$) do not depend on them:

$$[S^\alpha, S^0]_- = -\mathrm{i}J_{\alpha,\beta}[S^\beta, S^\gamma]_+ \qquad [S^\alpha, S^\beta]_- = \mathrm{i}[S^0, S^\gamma]_+ \tag{A.6}$$

where (α, β, γ) stands for any cyclic permutation of $(1, 2, 3)$, $[A, B]_\pm = AB \pm BA$, and $J_{\alpha,\beta} = (W_\alpha^2 - W_\beta^2)/(W_\gamma^2 - W_0^2)$, i.e.

$$J_{12} = \frac{\theta_{01}(\eta;\tau)^2\theta_{11}(\eta;\tau)^2}{\theta_{00}(\eta;\tau)^2\theta_{10}(\eta;\tau)^2}$$
$$J_{23} = \frac{\theta_{10}(\eta;\tau)^2\theta_{11}(\eta;\tau)^2}{\theta_{00}(\eta;\tau)^2\theta_{01}(\eta;\tau)^2}$$
$$J_{31} = -\frac{\theta_{00}(\eta;\tau)^2\theta_{11}(\eta;\tau)^2}{\theta_{01}(\eta;\tau)^2\theta_{10}(\eta;\tau)^2}.$$

The *spin ℓ representation* of the Sklyanin algebra, $\rho^\ell : U_{\tau,\eta}(sl(2)) \to \mathrm{End}_{\mathbb{C}}(\Theta_{00}^{4\ell})$ is defined as follows. The representation space is

$$\Theta_{00}^{4\ell+} = \{f(z) | f(z+1) = f(-z) = f(z), f(z+\tau) = \exp^{-4\ell\pi\mathrm{i}(2z+\tau)} f(z)\}. \tag{A.7}$$

It is easy to see that $\dim\Theta_{00}^{4\ell+} = 2\ell + 1$. The generators of the algebra act on this space as difference operators:

$$(\rho^\ell(S^a)f)(z) = \frac{s_a(z-\ell\eta)f(z+\eta) - s_a(-z-\ell\eta)f(z-\eta)}{\theta_{11}(2z;\tau)} \tag{A.8}$$

where

$$s_0(z) = \theta_{11}(\eta;\tau)\theta_{11}(2z;\tau) \qquad s_1(z) = \theta_{10}(\eta;\tau)\theta_{10}(2z;\tau)$$
$$s_2(z) = \mathrm{i}\theta_{00}(\eta;\tau)\theta_{00}(2z;\tau) \qquad s_3(z) = \theta_{01}(\eta;\tau)\theta_{01}(2z;\tau).$$

These representations reduce to the usual spin ℓ representations of $U(sl(2))$ for $J_{\alpha\beta} \to 0$ ($\eta \to 0$). In particular, in the case $\ell = 1/2$, the S^a are expressed by the Pauli matrices σ^a as follows. Take $(\theta_{00}(2z; 2\tau) - \theta_{10}(2z; 2\tau), \theta_{00}(2z; 2\tau) + \theta_{10}(2z; 2\tau))$ as a basis of Θ_{00}^{2+}. With respect to this basis the S^a have matrix forms

$$\rho^{1/2}(S^a) = 2\frac{\theta_{00}(\eta;\tau)\theta_{01}(\eta;\tau)\theta_{10}(\eta;\tau)\theta_{11}(\eta;\tau)}{\theta_{00}(0;\tau)\theta_{01}(0;\tau)\theta_{10}(0;\tau)}\sigma^a. \tag{A.9}$$

Since the relations (A.6) are homogeneous, an overall constant factor in a representation is not essential.

There are involutive automorphisms of the Sklyanin algebra $U_{\tau,\eta}(sl(2))$ found by Sklyanin [31]: for $a = 1, 2, 3$,

$$X_a : (S^0, S^a, S^b, S^c) \mapsto (S^0, S^a, -S^b, -S^c) \tag{A.10}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. Combining these operators with ρ^ℓ, we obtain another representation $\rho^\ell \circ X_a$ of $U_{\tau,\eta}(sl(2))$, but there is a unitary operator U_a intertwining ρ^ℓ and $\rho^\ell \circ X_a$ [31]:

$$U_1 : \Theta_{00}^{4\ell+} \ni f(z) \mapsto (U_1 f)(z) = \mathrm{e}^{\pi \mathrm{i}\ell} f\left(z + \frac{1}{2}\right)$$
$$U_3 : \Theta_{00}^{4\ell+} \ni f(z) \mapsto (U_3 f)(z) = \mathrm{e}^{\pi \mathrm{i}\ell} \mathrm{e}^{\pi \mathrm{i}\ell(4z+\tau)} f\left(z + \frac{\tau}{2}\right)$$

and $U_2 = U_3 U_1$. Direct calculations show that $X_a(\rho^\ell(S^b)) = U_a^{-1}\rho^\ell(S^b)U_a$. Operators U_a satisfy the relations: $U_a^2 = (-1)^{2\ell}$, $U_a U_b = (-1)^{2\ell} U_b U_a = U_c$.

Baxter's R-matrix (A.5) is a 4×4 matrix proportional to that in [37]:

$$R(\lambda; \mathrm{i}t) = \begin{pmatrix} a(\lambda) & 0 & 0 & d(\lambda) \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ d(\lambda) & 0 & 0 & a(\lambda) \end{pmatrix} \tag{A.11}$$

where functions a, b, c, d are defined by

$$\begin{aligned}
a(\lambda) &= C_1 \theta_{01}(2\mathrm{i}t\eta; 2\mathrm{i}t)\theta_{01}(\mathrm{i}t\lambda; 2\mathrm{i}t)\theta_{11}(\mathrm{i}t\lambda + 2\mathrm{i}t\eta; 2\mathrm{i}t)\\
b(\lambda) &= C_1 \theta_{11}(2\mathrm{i}t\eta; 2\mathrm{i}t)\theta_{01}(\mathrm{i}t\lambda; 2\mathrm{i}t)\theta_{01}(\mathrm{i}t\lambda + 2\mathrm{i}t\eta; 2\mathrm{i}t)\\
c(\lambda) &= C_1 \theta_{01}(2\mathrm{i}t\eta; 2\mathrm{i}t)\theta_{11}(\mathrm{i}t\lambda; 2\mathrm{i}t)\theta_{01}(\mathrm{i}t\lambda + 2\mathrm{i}t\eta; 2\mathrm{i}t)\\
d(\lambda) &= C_1 \theta_{11}(2\mathrm{i}t\eta; 2\mathrm{i}t)\theta_{11}(\mathrm{i}t\lambda; 2\mathrm{i}t)\theta_{11}(\mathrm{i}t\lambda + 2\mathrm{i}t\eta; 2\mathrm{i}t)\\
C_1 &= \frac{-2\exp(-\pi t\lambda(\lambda + 2\eta))}{\theta_{01}(0; 2\mathrm{i}t)\theta_{01}(2\mathrm{i}t\eta; 2\mathrm{i}t)\theta_{11}(2\mathrm{i}t\eta; 2\mathrm{i}t)}.
\end{aligned}$$

Obviously eigenvalues of this matrix are

$$\begin{aligned}
a(\lambda) + d(\lambda) &= C_2 \frac{\theta_{00}\left(\frac{\mathrm{i}t\lambda}{2} - \mathrm{i}t\eta; \mathrm{i}t\right)}{\theta_{00}\left(\frac{\mathrm{i}t\lambda}{2} + \mathrm{i}t\eta; \mathrm{i}t\right)} &\qquad a(\lambda) - d(\lambda) &= C_2 \frac{\theta_{01}\left(\frac{\mathrm{i}t\lambda}{2} - \mathrm{i}t\eta; \mathrm{i}t\right)}{\theta_{01}\left(\frac{\mathrm{i}t\lambda}{2} + \mathrm{i}t\eta; \mathrm{i}t\right)}\\
b(\lambda) + c(\lambda) &= C_2 \frac{\theta_{10}\left(\frac{\mathrm{i}t\lambda}{2} - \mathrm{i}t\eta; \mathrm{i}t\right)}{\theta_{10}\left(\frac{\mathrm{i}t\lambda}{2} + \mathrm{i}t\eta; \mathrm{i}t\right)} &\qquad b(\lambda) - c(\lambda) &= C_2 \frac{\theta_{11}\left(\frac{\mathrm{i}t\lambda}{2} - \mathrm{i}t\eta; \mathrm{i}t\right)}{\theta_{11}\left(\frac{\mathrm{i}t\lambda}{2} + \mathrm{i}t\eta; \mathrm{i}t\right)}
\end{aligned} \tag{A.12}$$

where

$$C_2 = 2\mathrm{e}^{-\pi t\lambda(\lambda+2\eta)} \frac{\theta_{00}\left(\frac{\mathrm{i}t\lambda}{2} + \mathrm{i}t\eta; \mathrm{i}t\right)\theta_{01}\left(\frac{\mathrm{i}t\lambda}{2} + \mathrm{i}t\eta; \mathrm{i}t\right)\theta_{10}\left(\frac{\mathrm{i}t\lambda}{2} + \mathrm{i}t\eta; \mathrm{i}t\right)\theta_{11}\left(\frac{\mathrm{i}t\lambda}{2} + \mathrm{i}t\eta; \mathrm{i}t\right)}{\theta_{00}(\mathrm{i}t\eta; \mathrm{i}t)\theta_{01}(\mathrm{i}t\eta; \mathrm{i}t)\theta_{10}(\mathrm{i}t\eta; \mathrm{i}t)\theta_{11}(\mathrm{i}t\eta; \mathrm{i}t)}.$$

Appendix B. Proof of the sum rule

We prove here the sum rule of λ_j's, theorem 1.4.1. See section 1.3 for notation.

Let us introduce a determinant $t^r(\lambda)$ of a $(r-1) \times (r-1)$ matrix, elements of which are defined by

(1) $(j, j+1)$-elements $= h(\lambda + 2(j - \ell)\eta)$;

(2) (j, j)-elements $= t(\lambda + 2j\eta)$;

(3) $(j, j-1)$-elements $= h(\lambda + 2(j + \ell)\eta)$;

(4) other elements are 0,

where $t(\lambda)$ is the eigenvalue of the transfer matrix $T(\lambda)$ on the Bethe vector $\Psi_\nu(\lambda_1, \ldots, \lambda_M)$ and $h(z) = (2\theta_{11}(z))^N$. (This determinant is related to a fused model.) Since $t(\lambda)$ is an entire function of λ (recall that the transfer matrix $T(\lambda)$ itself is an entire function of λ), $t^r(\lambda)$ is an entire function of λ. (In other words, the analyticity is a consequence of the Bethe equations as noted in section 1.3.) Our third assumption (see theorem 1.4.1) is

(iii) *$t^r(\lambda)$ is not identically zero.*

Let us define $\tilde{t}^r(\lambda)$ by

$$\tilde{t}^r(\lambda) := Q(\lambda+2\eta)Q(\lambda+4\eta)\dots Q(\lambda+2(r-1)\eta)t^r(\lambda). \tag{B.1}$$

Then because of (1.3.12)

$$t(\lambda) := h(\lambda+2\ell\eta)\frac{Q(\lambda-2\eta)}{Q(\lambda)} + h(\lambda-2\ell\eta)\frac{Q(\lambda+2\eta)}{Q(\lambda)}$$

function $\tilde{t}^r(\lambda)$ is expressed as a determinant of a matrix such that

(1) $(j, j+1)$-element $= a_+(\lambda+2j\eta)$;

(2) (j, j)-element $= a_-(\lambda+2j\eta) + a_+(\lambda+2j\eta)$;

(3) $(j, j-1)$-element $= a_-(\lambda+2j\eta)$;

(4) other elements are 0,

where

$$a_-(\lambda) = h(\lambda+2\ell\eta)Q(\lambda-2\eta)$$
$$a_+(\lambda) = h(\lambda-2\ell\eta)Q(\lambda+2\eta).$$

This determinant can be easily expanded, the result being

$$\begin{aligned}\tilde{t}^r(\lambda) &= \sum_{j=1}^{r} a_-(\lambda+2\eta)\dots a_-(\lambda+2(j-1)\eta)a_+(\lambda+2j\eta)\dots a_+(\lambda+2(r-1)\eta)\\ &= h(\lambda+2(\ell+1)\eta)\dots h(\lambda+2(r-\ell-1)\eta)Q(\lambda)(f_0(\lambda)+\dots+f_{r-1}(\lambda))\end{aligned} \tag{B.2}$$

where

$$f_k(\lambda) = \prod_{j=1}^{2\ell} h(\lambda+2(k-\ell+j)\eta)\prod_{j=1}^{k-1} Q(\lambda+2j\eta)\prod_{j=k+2}^{r} Q(\lambda+2j\eta).$$

By definition (1.3.11), $Q(\lambda)$ has automorphic property:

$$\begin{aligned}Q(\lambda+1) &= (-1)^{N\ell-\nu}Q(\lambda)\\ Q(\lambda+\tau) &= \mathrm{e}^{-\pi\mathrm{i}N\ell(1+\tau+2\lambda)-\pi\mathrm{i}\tau\nu+2\pi\mathrm{i}\sum_{j=1}^{M}\lambda_j}Q(\lambda).\end{aligned} \tag{B.3}$$

Hence $f_k(\lambda+2\eta) = f_{k+1}(\lambda)$ $(f_r(\lambda) = f_0(\lambda))$ and $F(\lambda) = f_0(\lambda)+\dots+f_{r-1}(\lambda)$ has a period 2η: $F(\lambda+2\eta) = F(\lambda)$.

Now we proceed in four steps.

Step 1. First we show that $Q(\lambda)F(\lambda)/Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta)$ is an entire function of λ. Since

$$\begin{aligned}t^r(\lambda) &= \frac{\tilde{t}^r(\lambda)}{Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta)}\\ &= h(\lambda+2(\ell+1)\eta)\dots h(\lambda+2(r-\ell-1)\eta)\frac{Q(\lambda)F(\lambda)}{Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta)}\end{aligned} \tag{B.4}$$

is an entire function of λ, we have only to show that any zero of the denominator is not a zero of $h(\lambda+2(\ell+1)\eta)\dots h(\lambda+2(r-\ell-1)\eta)$. Zeros of $h(\lambda)$ is $0 \bmod \mathbb{Z}+\mathbb{Z}\tau$. Hence the last statement is true if assumption (i) of theorem 1.4.1 is fulfilled.

Step 2. We show that $F(\lambda)/Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta)$ is an entire function of λ.

As a consequence of step 1, we know that the only possible poles of $F(\lambda)/Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta)$ exist at zeros of $Q(\lambda)$. Suppose λ_j is a pole of $F(\lambda)/Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta)$. Then

$$\mathrm{ord}_{\lambda_j}F(\lambda) < \mathrm{ord}_{\lambda_j}(Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta)) \leqslant \mathrm{ord}_{\lambda_j}F(\lambda)+\mathrm{ord}_{\lambda_j}Q(\lambda). \tag{B.5}$$

Here ord_{λ_j} is the order of zero at λ_j. Assumption (ii) of theorem 1.4.1 says that there is an integer a such that $\mathrm{ord}_{\lambda_j+2a\eta}Q(\lambda)=0$. On the other hand periodicity $F(\lambda+2\eta)=F(\lambda)$ implies that $\mathrm{ord}_{\lambda_j+2a\eta}F(\lambda)=\mathrm{ord}_{\lambda_j}F(\lambda)$. Therefore

$$\begin{aligned}
\mathrm{ord}_{\lambda_j+2a\eta}&(Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta))\\
&=\mathrm{ord}_{\lambda_j}(Q(\lambda+2(a+1)\eta)\dots Q(\lambda+2(r+a-1)\eta))\\
&=\mathrm{ord}_{\lambda_j}(Q(\lambda+2a\eta)\dots Q(\lambda+2(r+a-1)\eta))\\
&=\mathrm{ord}_{\lambda_j}(Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta)) > \mathrm{ord}_{\lambda_j}F(\lambda)\\
&=\mathrm{ord}_{\lambda_j+2a\eta}F(\lambda)\\
&=\mathrm{ord}_{\lambda_j+2a\eta}Q(\lambda)F(\lambda).
\end{aligned} \tag{B.6}$$

Here we used (B.3) and (B.5). This inequality means that $Q(\lambda)F(\lambda)/(Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta))$ has a pole at $\lambda_j+2a\eta$. This contradicts step 1.

Step 3. Now we show that even $F(\lambda)/(Q(\lambda)Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta))$ *is an entire* function of λ. It follows from step 2 that for any $j=0,1,\dots,r-1$

$$\frac{Q(\lambda+2j\eta)F(\lambda)}{Q(\lambda)Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta)}$$

is an entire function. Suppose $F(\lambda)/(Q(\lambda)Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta))$ has a pole at λ_0. Then λ_0 should be a zero of $Q(\lambda+2j\eta)$, $j=0,\dots,r-1$. Taking (B.3) into account, this contradicts assumption (ii).

Step 4. We have shown that $F(\lambda)/G(\lambda)$ is an entire function where $G(\lambda)=Q(\lambda)Q(\lambda+2\eta)\dots Q(\lambda+2(r-1)\eta)$. Using (B.3) and

$$h(\lambda+1)=(-1)^N h(\lambda)$$
$$h(\lambda+\tau)=\mathrm{e}^{-\pi\mathrm{i}N(1+\tau)-2\pi\mathrm{i}\lambda}h(\lambda)$$

we obtain

$$\frac{F(\lambda+1)}{G(\lambda+1)}=\frac{F(\lambda)}{G(\lambda)} \tag{B.7}$$

$$\frac{F(\lambda+\tau)}{G(\lambda+\tau)}=\exp[2\pi\mathrm{i}(\nu\tau-2\sum_{j=1}^{M}\lambda_j)]\frac{F(\lambda)}{G(\lambda)}. \tag{B.8}$$

Holomorphy of F/G and periodicity (B.7) make it possible to expand F/G into a Fourier series:

$$(F/G)(\lambda)=\sum_{n\in\mathbb{Z}}(F/G)_n\mathrm{e}^{2\pi\mathrm{i}n\lambda}.$$

Substituting $\lambda+\tau$ *into this expansion and comparing with* (B.8), *we find that each* coefficient should satisfy

$$(F/G)_n=(F/G)_n\exp[2\pi\mathrm{i}((\nu-n)\tau-2\sum_{j=1}^{M}\lambda_j)].$$

Since $\operatorname{Im}\tau > 0$, there exists only one $n = n_2$ such that $(F/G)_{n_2} \neq 0$ and it satisfies $(\nu - n_2)\tau - 2\sum_{j=1}^{M}\lambda_j =: -n_0 \in \mathbb{Z}$. Putting $n_1 = \nu - n_2$, we have $2\sum_{j=1}^{M}\lambda_j = n_0 + n_1\tau$.

It follows from the above argument that $t^r(\lambda)$ has the following form with a suitable integer n:

$$t^r(\lambda) = \text{constant}\ e^{2\pi\, i n\lambda} h(\lambda + 2(\ell+1)\eta)\dots h(\lambda + 2(r-\ell-1)\eta)Q(\lambda)^2. \tag{B.9}$$

Appendix C. Table of useful functions

Here we collect properties of functions used in chapter II.

Logarithm of quotient of theta functions. A function Φ defined by (2.2.2),

$$\Phi(x; i\mu t) = \frac{1}{i}\log\frac{\theta_{11}(x + i\mu t; it)}{\theta_{11}(x - i\mu t; it)} + \pi$$

has the following Fourier expansion if $0 < \mu < 1/2$:

$$\Phi(x; i\mu t) = -2\pi x - 2\sum_{n=1}^{\infty}\frac{\sin h\pi n(1-2\mu)t}{n\sin h\pi nt}\sin 2\pi nx. \tag{C.1}$$

Hence

$$\frac{d}{dx}\Phi(x; i\mu t) = -2\pi\left(1 + 2\sum_{n=1}^{\infty}\frac{\sin h\pi n(1-2\mu)t}{\sin h\pi nt}\cos 2\pi nx\right). \tag{C.2}$$

A function Ψ defined by (2.3.29),

$$\Psi(x; i\mu t) = \frac{1}{i}\log\frac{\theta_{01}(x + i\mu t; it)}{\theta_{01}(x - i\mu t; it)}$$

has the following Fourier expansion if $0 < \mu < 1/2$:

$$\Psi(x; i\mu t) = 2\sum_{n=1}^{\infty}\frac{\sin h2\pi n\mu t}{n\sin h\pi nt}\sin 2\pi nx. \tag{C.3}$$

Hence

$$\frac{d}{dx}\Psi(x; i\mu t) = 4\pi\sum_{n=1}^{\infty}\frac{\sin h2\pi n\mu t}{\sin h\pi nt}\cos 2\pi nx. \tag{C.4}$$

Lemma C.2. For $0 < a < b$, a series

$$\sum_{n\in\mathbb{Z}}\frac{\sin h\pi na}{\sin h\pi nb}e^{2\pi\, inx}$$

is positive for $x \in \mathbb{R}$. Here the term $n = 0$ is understood as a/b.

Proof. Define a function $f(y; x)$ by

$$f(y; x) = \frac{\sin h\pi ya}{\sin h\pi yb}e^{2\pi\, iyx}$$

$f(0; x) = a/b$. The Fourier transformation of this function is

$$\begin{aligned}\hat{f}(\xi; x) &= \int_{-\infty}^{\infty} f(y; x)e^{-2\pi\, iy\xi}\,dy \\ &= \frac{2\sin(a\pi/b)}{b}\frac{e^{-2\pi|x-\xi|/b}}{(e^{-2\pi|x-\xi|/b} + \cos(a\pi/b))^2 + \sin^2(a\pi/b)} > 0.\end{aligned} \tag{C.5}$$

By Poisson's summation formula we have

$$\sum_{n\in\mathbb{Z}} \frac{\sin h\pi na}{\sin h\pi nb} e^{2\pi i n x} = \sum_{n\in\mathbb{Z}} f(n;x) = \sum_{n\in\mathbb{Z}} \hat{f}(n;x) > 0.$$

This proves the lemma. □

For example $d/dx(\Phi(x; i\mu t)) < 0$, because of lemma C.2.

q-Γ function. A q-analogue of the Γ function is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \tag{C.6}$$

where $(x;q)_\infty = \prod_{n=0}^{\infty}(1-xq^n)$.

Double infinite product $(x;q_1,q_2)_\infty$ is defined by $(x;q_1,q_2)_\infty = \prod_{n_1,n_2=0}^{\infty}(1-xq_1^{n_1}q_2^{n_2})$. If $x+y=z+w$, the following relation holds:

$$\frac{(q^x;q^a,q)_\infty(q^y;q^a,q)_\infty}{(q^z;q^a,q)_\infty(q^w;q^a,q)_\infty} = \prod_{n=1}^{\infty}\frac{\Gamma_q(z+an)\Gamma_q(w+an)}{\Gamma_q(x+an)\Gamma_q(y+an)}. \tag{C.7}$$

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